Geometry of Black Holes
revised July 2018

Piotr T. Chruściel
University of Vienna

piotr.chrusciel@univie.ac.at

http://homepage.univie.ac.at/piotr.chrusciel

August 15, 2018

Contents

Contents **iii**

I Black holes **1**

1 An introduction to black holes **3**
1.1 Black holes as astrophysical objects . . . 3
1.2 The Schwarzschild solution and its extensions . . . 9
1.2.1 The singularity $r = 0$ . . . 14
1.2.2 Eddington-Finkelstein extension . . . 14
1.2.3 The Kruskal-Szekeres extension . . . 18
1.2.4 Other coordinate systems, higher dimensions . . . 23
1.2.5 Some geodesics . . . 29
1.2.6 The Flamm paraboloid . . . 31
1.2.7 Fronsdal's embedding . . . 33
1.2.8 Conformal Carter-Penrose diagrams . . . 34
1.2.9 Weyl coordinates . . . 36
1.3 Some general notions . . . 37
1.3.1 Isometries . . . 37
1.3.2 Killing horizons . . . 38
1.3.3 Surface gravity . . . 40
1.3.4 The orbit-space geometry near Killing horizons . . . 47
1.3.5 Near-horizon geometry . . . 48
1.3.6 Asymptotically flat stationary metrics . . . 52
1.3.7 Domains of outer communications, event horizons . . . 54
1.4 Extensions . . . 55
1.4.1 Distinct extensions . . . 55
1.4.2 Inextendibility . . . 56
1.4.3 Uniqueness of a class of extensions . . . 58
1.5 The Reissner-Nordström metrics . . . 61
1.6 The Kerr metric . . . 63
1.6.1 Non-degenerate solutions ($a^2 < m^2$): Bifurcate horizons . 72
1.6.2 Surface gravity, thermodynamical identities . . . 76
1.6.3 Carter's time machine . . . 78
1.6.4 Extreme case $a^2 = m^2$: horizon, near-horizon geometry, cylindrical ends . . . 79

1.6.5 The Ernst map for the Kerr metric . . . 81
1.6.6 The orbit space metric . . . 82
1.6.7 Kerr-Schild coordinates . . . 83
1.6.8 Dain coordinates . . . 83
1.7 Majumdar-Papapetrou multi black holes . . . 84
1.7.1 Adding bifurcation surfaces . . . 88
1.8 The Kerr-de Sitter/Kerr-anti-de Sitter metric . . . 89
1.8.1 Asymptotic behavior . . . 92

2 Emparan-Reall “black rings” 95
2.1 $x \in \{\xi_1, \xi_2\}$ . . . 97
2.2 Signature . . . 98
2.3 $y = \xi_1$ . . . 99
2.4 Asymptotic flatness . . . 99
2.5 $y \to \pm\infty$ . . . 102
2.6 Ergoregion . . . 103
2.7 Black ring . . . 104
2.8 Some further properties . . . 105
2.9 A Kruskal-Szekeres type extension . . . 110
2.10 Global structure . . . 113
2.10.1 The event horizon has $S^2 \times S^1 \times \mathbb{R}$ topology . . . 114
2.10.2 Inextendibility at $z = \xi_F$, maximality . . . 115
2.10.3 Conformal infinity $\mathscr{I}$ . . . 116
2.10.4 Uniqueness and non-uniqueness of extensions . . . 117
2.10.5 Other coordinate systems . . . 118

3 Rasheed’s Kaluza-Klein black holes 121
3.1 Rasheed’s metrics . . . 121
3.2 Zeros of the denominators . . . 123
3.3 Regularity at the outer Killing horizon $\mathcal{H}_+$ . . . 126
3.4 Asymptotic behaviour . . . 127

4 Diagrams, extensions 129
4.1 Causality for a class of bloc-diagonal metrics . . . 129
4.1.1 Riemannian aspects . . . 130
4.1.2 Causality . . . 132
4.2 The building blocs . . . 133
4.2.1 Two-dimensional Minkowski spacetime . . . 133
4.2.2 Higher dimensional Minkowski spacetime . . . 135
4.2.3 $\int F^{-1}$ diverging at both ends . . . 137
4.2.4 $\int F^{-1}$ diverging at one end only . . . 139
4.2.5 Generalised Kottler metrics with $\Lambda < 0$ and $m = 0$ . . . 140
4.3 Putting things together . . . 141
4.3.1 Four-blocs gluing . . . 141
4.3.2 Two-blocs gluing . . . 146
4.4 General rules . . . 147
4.5 Black holes / white holes . . . 149

4.6 Birmingham metrics . . . 150
4.6.1 Cylindrical solutions . . . 151
4.6.2 Naked singularities . . . 152
4.6.3 Spatially periodic time-symmetric initial data . . . 153
4.6.4 Killing horizons . . . 153
4.6.5 Curvature . . . 154
4.6.6 The Euclidean Schwarzschild - anti de Sitter metric . . . 158
4.7 Projection diagrams . . . 162
4.7.1 The definition . . . 163
4.7.2 Simplest examples . . . 165
4.7.3 The Kerr metrics . . . 167
4.7.4 The Kerr-Newman metrics . . . 176
4.7.5 The Kerr - de Sitter metrics . . . 177
4.7.6 The Kerr-Newman - de Sitter metrics . . . 181
4.7.7 The Kerr-Newman - anti de Sitter metrics . . . 184
4.7.8 The Emparan-Reall metrics . . . 186
4.7.9 The Pomeransky-Senkov metrics . . . 191
4.8 Black holes and Cauchy horizons . . . 193
4.8.1 Kerr-Newman-(a)dS-type and Pomeransky-Senkov-type models . . . 194
4.8.2 Taub-NUT metrics . . . 196

5 Alternative approaches 201
5.1 The standard approach and its shortcomings . . . 201
5.2 Black holes without Scri . . . 205
5.2.1 Naive black holes . . . 206
5.2.2 Quasi-local black holes . . . 208

6 Dynamical black holes: the Robinson-Trautman metrics 213
6.1 Robinson–Trautman spacetimes. . . . 213
6.1.1 $m > 0$ . . . 216
6.1.2 $m < 0$ . . . 219
6.1.3 $\Lambda \neq 0$ . . . 220

II Background Material 225

A Pseudo-Riemannian geometry 227
A.1 Manifolds . . . 227
A.2 Scalar functions . . . 228
A.3 Vector fields . . . 228
A.3.1 Lie bracket . . . 231
A.4 Covectors . . . 231
A.5 Bilinear maps, two-covariant tensors . . . 233
A.6 Tensor products . . . 234
A.6.1 Contractions . . . 236
A.7 Raising and lowering of indices . . . 236

A.8 The Lie derivative . . . 238
A.8.1 A pedestrian approach . . . 238
A.8.2 The geometric approach . . . 241
A.9 Covariant derivatives . . . 247
A.9.1 Functions . . . 248
A.9.2 Vectors . . . 249
A.9.3 Transformation law . . . 250
A.9.4 Torsion . . . 251
A.9.5 Covectors . . . 251
A.9.6 Higher order tensors . . . 253
A.10 The Levi-Civita connection . . . 253
A.10.1 Geodesics and Christoffel symbols . . . 255
A.11 “Local inertial coordinates” . . . 256
A.12 Curvature . . . 258
A.12.1 Bianchi identities . . . 262
A.12.2 Pair interchange symmetry . . . 265
A.12.3 Summmary for the Levi-Civita connection . . . 267
A.12.4 Curvature of product metrics . . . 268
A.12.5 An identity for the Riemann tensor . . . 269
A.13 Geodesics . . . 270
A.14 Geodesic deviation (Jacobi equation) . . . 272
A.15 Exterior algebra . . . 274
A.16 Submanifolds, integration, and Stokes’ theorem . . . 278
A.16.1 Hypersurfaces . . . 279
A.17 Odd forms (densities) . . . 282
A.18 Moving frames . . . 283
A.19 Lovelock Theorems . . . 292
A.19.1 Lovelock Lagrangeans . . . 293
A.19.2 Lovelock tensors . . . 296
A.20 Clifford algebras . . . 299
A.20.1 Eigenvalues of γ-matrices . . . 304
A.21 Killing vectors and isometries . . . 305
A.21.1 Killing vectors . . . 306
A.21.2 Analyticity of isometries . . . 310
A.21.3 The structure of isometry groups of asymptotically flat spacetimes . . . 310
A.21.4 Killing vectors vs. isometry groups . . . 311
A.22 Null hyperplanes . . . 313
A.23 The geometry of null hypersurfaces . . . 316
A.24 Elements of causality theory . . . 319

B A collection of identities 321
B.1 ADM notation . . . 321
B.2 Some commutators . . . 321
B.3 Bianchi identities . . . 322
B.4 Linearisations . . . 322
B.5 Warped products . . . 322

B.6 Hypersurfaces 323
B.7 Conformal transformations 323
B.8 Laplacians on tensors 324
B.9 Stationary metrics 325

Bibliography **327**

Part I

Black holes

Chapter 1

An introduction to black holes

Black holes belong to the most fascinating objects predicted by Einstein's theory of gravitation. Although they have been studied for years,[1] they still attract tremendous attention in the physics and astrophysics literature. It turns out that several field theories are known to possess solutions which exhibit black hole properties:

- The "standard" gravitational ones which, according to our current postulates, are black holes for all classical fields.
- The "dumb holes", which are the sonic counterparts of black holes, first discussed by Unruh [269].
- The "optical" ones – the black-hole counterparts arising in the theory of moving dielectric media, or in non-linear electrodynamics [186, 221].
- The "numerical black holes" – objects constructed by numerical general relativists.

(An even longer list of models and submodels can be found in [13].) In this work we shall discuss various aspects of the above. The reader is referred to [34, 105, 155, 162, 232, 272] and references therein for a review of quantum aspects of black holes.

Insightful animations of journeys in a black hole spacetime can be found at `http://jilawww.colorado.edu/~ajsh/insidebh/schw.html`.

We start with a short review of the observational status of black holes in astrophysics.

1.1 Black holes as astrophysical objects

When a star runs out of nuclear fuel, it must find ways to fight gravity. Current physics predicts that dead stars with masses up to the Chandrasekhar limit, $M_{\rm mcH} = 1.4 M_\odot$, become white dwarfs, where electron degeneracy supplies the

[1] The reader is referred to the introduction to [48] for an excellent concise review of the history of the concept of a black hole, and to [47, 160] for more detailed ones.

necessary pressure. Above the Chandrasekhar limit $1.4M_\odot$, and up to a second mass limit, $M_{\mathrm{NS,max}} \sim 2 - 3M_\odot$, dead stars are expected to become neutron stars, where neutron degeneracy pressure holds them up. If a dead star has a mass $M > M_{\mathrm{NS,max}}$, there is no known force that can hold the star up. What we have then is a black hole.

While there is growing evidence that black holes do indeed exist in astrophysical objects, and that alternative explanations for the observations discussed below seem less convincing, it should be borne in mind that no undisputed evidence of occurrence of black holes has been presented so far. The flagship black hole candidate used to be Cygnus X-1, known and studied for years (*cf., e.g.,* [48, 226]), and it still remains a strong one. Table 1.1[2] lists a series of further strong black hole candidates in X-ray binary systems; M_c is mass of the compact object and M_* is that of its optical companion; some other candidates, as well as references, can be found in [40, 203, 212, 215]. The binaries have been divided into two families: the High Mass X-ray Binaries (HMXB), where the companion star is of (relatively) high mass, and the Low Mass X-ray Binaries (LMXB), where the companion is typically below a solar mass. The LMXB's include the "X-ray transients", so-called because of flaring-up behaviour. This particularity allows to make detailed studies of their optical properties during the quiescent periods, which would be impossible during the periods of intense X-ray activity. The stellar systems listed have X-ray spectra which are neither periodic (that would correspond to a rotating neutron star), nor recurrent (which is interpreted as thermonuclear explosions on a neutron star's hard surface). The final selection criterion is that of the mass M_c exceeding the Chandrasekhar limit $M_C \approx 3$ solar masses $M_\odot$.[3] According to the authors of [48], the strongest stellar-mass black hole candidate in 1999 was V404 Cygni, which belongs to the LMXB class. Table 1.1 should be put into perspective by realizing that, by some estimates [193], a typical galaxy – such as ours – should harbour $10^7 - 10^8$ stellar black mass holes. We note an interesting proposal, put forward in [49], to carry out observations by gravitational microlensing of some 20 000 stellar-mass black holes that are predicted [206] to cluster within 0.7 pc of Sgr A* (the centre of our galaxy).

It is now widely accepted that quasars and active galactic nuclei are powered by accretion onto massive black holes [116, 195, 281]. Further, over the last few years there has been increasing evidence that massive dark objects may reside at the centres of most, if not all, galaxies [194, 242]. In several cases the best explanation for the nature of those objects is that they are "heavyweight" black holes, with masses ranging from 10^6 to 10^{10} solar masses. Table 1.2[4] lists some supermassive black hole candidates; some other candidates, as well as precise references, can be found in [173, 203, 204, 241]. The main criterion for finding candidates for such black holes is the presence of a large mass within a small region; this is determined by maser line spectroscopy, gas spectroscopy, or by

[2] The review [201] lists forty binaries containing a black hole candidate.

[3] See [203] for a discussion and references concerning the value of M_C.

[4] The table lists those galaxies which are listed both in [204] and [173]; we note that some candidates from earlier lists [241] do not occur any more in [173, 204]. Nineteen of the observations listed have been published in 2000 or 2001.

Table 1.1: Stellar mass black hole candidates (from [193])

Type	Binary system	$M_c/M_\odot$	$M_*/M_\odot$
HMXB:	Cygnus X-1	11–21	24–42
	LMC X-3	5.6 –7.8	20
	LMC X-1	≥ 4	4–8
LMXB:	V 404 Cyg	10–15	≈ 0.6
	A 0620-00	5–17	0.2–0.7
	GS 1124-68 (Nova Musc)	4.2–6.5	0.5–0.8
	GS 2000+25 (Nova Vul 88)	6-14	≈ 0.7
	GRO J 1655-40	$4.5 - 6.5$	≈ 1.2
	H 1705-25 (Nova Oph 77)	5–9	≈ 0.4
	J 04224+32	6–14	$\approx 0.3 - 0.6$

measuring the motion of stars orbiting around the galactic nucleus.

There seems to be consensus [173, 204, 216, 242] that the two most convincing supermassive black hole candidates are the galactic nuclei of NGC 4258 and of our own Milky Way [135, 139]. The determination of mass of the galactic nuclei via direct measurements of star motions has been made possible both by the unprecedentedly high angular resolution and sensitivity of the Hubble Space Telescope (HST), see also Figure 1.1.1, and by the adaptive-optics Keck Telescopes [280].

The reader is referred to [211] for a discussion of the maser emission lines and their analysis for the supermassive black hole candidate NGC 4258. An example of measurements via gas spectrography is given by the analysis of the HST observations of the radio galaxy M 87 [268] (compare [195]): A spectral analysis shows the presence of a disk-like structure of ionized gas in the innermost few arc seconds in the vicinity of the nucleus of M 87. The velocity of the gas measured by spectroscopy (cf. Figure 1.1.2) at a distance from the nucleus of the order of 6×10^{17} m, shows that the gas recedes from us on one side, and approaches us on the other, with a velocity difference of about 920 km s^{-1} . This leads to a mass of the central object of $\sim 3 \times 10^9$ $M_\odot$, and no known form of matter with this mass is likely to occupy such a (relatively) small region except for a black hole. Figure 1.1.3 shows another image, reconstructed out of HST observations, of a recent candidate for a supermassive black hole – the (active) galactic nucleus of NGC 4438 [168].

There have been suggestions for existence for an intermediate-mass black hole orbiting three light-years from Sagittarius A*. This black hole of 1,300 solar masses is within a cluster of seven stars, possibly the remnant of a massive star

Table 1.2: Twenty-nine supermassive black hole candidates (from [173, 204])

dynamics of	host galaxy	$M_h/M_\odot$	host galaxy	$M_h/M_\odot$
water maser discs:	NGC 4258	4×10^7		
gas discs:	IC 1459	2×10^8	M 87	3×10^9
	NGC 2787	4×10^7	NGC 3245	2×10^8
	NGC 4261	5×10^8	NGC 4374	4×10^8
	NGC 5128	2×10^8	NGC 6251	6×10^8
	NGC 7052	3×10^8		
stars:	NGC 821	4×10^7	NGC 1023	4×10^7
	NGC 2778	1×10^7	NGC 3115	1×10^9
	NGC 3377	1×10^8	NGC 3379	1×10^8
	NGC 3384	1×10^7	NGC 3608	1×10^8
	NGC 4291	2×10^8	NGC 4342	3×10^8
	NGC 4473	1×10^8	NGC 4486B	5×10^8
	NGC 4564	6×10^7	NGC 4649	2×10^9
	NGC 4697	2×10^8	NGC 4742	1×10^7
	NGC 5845	3×10^8	NGC 7457	4×10^6
	Milky Way	3.7×10^6		

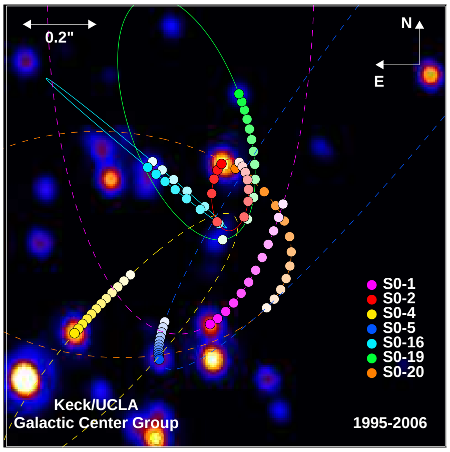

Figure 1.1.1: The orbits of stars within the central 1.0 × 1.0 arcseconds of our Galaxy. In the background, the central portion of a diffraction-limited image taken in 2006 is displayed. While every star in this image has been seen to move over the past 12 years, estimates of orbital parameters are only possible for the seven stars that have had significant curvature detected. The annual average positions for these seven stars are plotted as colored dots, which have increasing color saturation with time. Also plotted are the best fitting simultaneous orbital solutions. These orbits provide the best evidence yet for a supermassive black hole, which has a mass of 3.7 million times the mass of the Sun. The image was created by Andrea Ghez and her research team at UCLA, from data sets obtained with the W. M. Keck Telescopes, and is available at `http://www.astro.ucla.edu/~ghezgroup/gc/pictures/`.

cluster that has been stripped down by the Galactic Centre [196]. See [144] for a list of further intermediate-mass candidates.

A new twist to the observations of black holes has been added by the first direct detection of a gravitational wave in September 2015 [2], with a second wave observation in December 2015 [1] and a third one in January 2017 [3].

While there is widespread consensus that the waves have been detected by now, some scientific scepticism is in order. The observation requires the extraction of an absurdly small signal from overwhelmingly noisy data using sophisticated data analysis techniques. Even though the scientists working on the problem have made many efforts to ensure the validity of the claim, there always remains the possibility of instrumental, interpretational, or data analysis errors; see e.g. [97]. One needs also to keep in mind the possibility that the interpretation of the waves, as originating from black hole mergers, might be flawed. In any case there is strong evidence for a direct observation of gravitational waves now, and we can only hope that this evidence will keep growing stronger.

Having said this, the first event, christened GW150914 (for “Gravitation

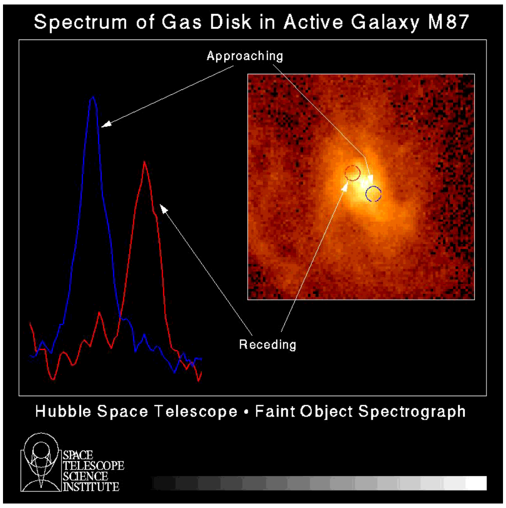

Figure 1.1.2: Hubble Space Telescope observations of spectra of gas in the vicinity of the nucleus of the radio galaxy M 87, NASA and H. Ford (STScI/JHU) [258].

Wave observed on September 14, 2015"), is thought to have been created by two black-holes with respective masses $36^{+5}_{-4}M_\odot$ and $29^{+4}_{-4}M_\odot$, merging into a final black hole with mass $62^{+4}_{-4}M_\odot$. An astounding $3^{+.5}_{-.5}M_\odot c^2$ amount of energy has been released within a fraction of a second into gravitational waves. The signal observed can be seen in Figure 1.1.4, p. 10.

The second event GW151226, illustrated by Figure 1.1.5, p. 11, is interpreted as representing the merger of two black holes of respective masses $14^{+8.3}_{-3.7}M_\odot$ and $7.5^{+2.3}_{-2.3}M_\odot$, leading to a final black holes of mass $21^{+5.9}_{-1.9}M_\odot$. Inspection of Figures 1.1.4 and 1.1.5 reveals that the GW151226 signal is nowhere as striking as GW150914, with a maximal amplitude smaller than the residual noise. Nevertheless, the estimated probability of a false detection for GW151226 is smaller than the convincingly small number 10^{-7}.

The third event GW170104, with wave forms displayed in Figure 1.1.6, p. 12, is thought to be created by the merger of two black holes of respective masses $31^{+8.6}_{-6}M_\odot$ and $20^{+5}_{-6}M_\odot$. The signal is somewhat reminiscent of that of GW150914, compare Figure 1.1.4.

The LIGO events give thus the first evidence of existence of black hole binaries, and of black holes with masses in the $10M_\odot - 100M_\odot$ range. The spectrum of lightweight-to-middleweight black holes, as known in early 2018, is illustrated in Figure 1.1.7

A compilation of black hole candidates as of 2004, some very tentative, can be found at `http://www.johnstonsarchive.net/relativity/bhctable.`

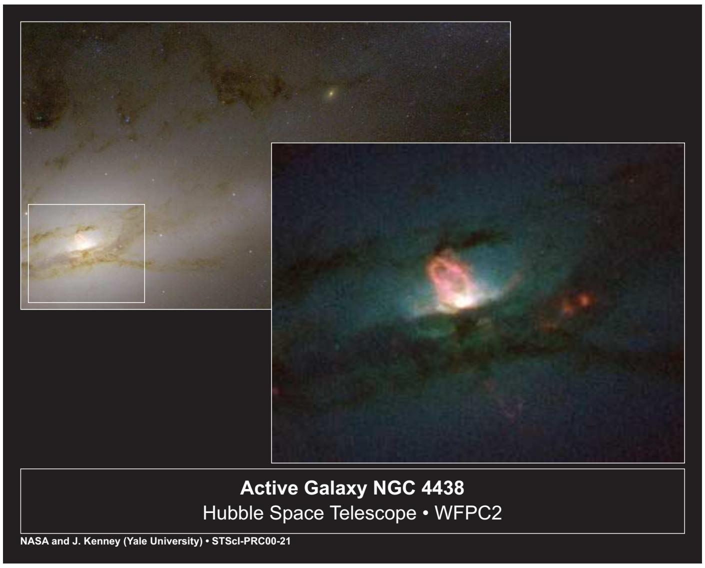

Figure 1.1.3: Hubble Space Telescope observations [168] of the nucleus of the galaxy NGC 4438, from the STScI Public Archive [258].

`html`. A maintained list can be found on Wikipedia, `https://en.wikipedia.org/wiki/List_of_black_holes`, it needs to be interpreted with the usual care.

We close this section by pointing out the review paper [41] which discusses both theoretical and experimental issues concerning *primordial* black holes.

1.2 The Schwarzschild solution and its extensions

Stationary solutions are of interest for a variety of reasons. As models for compact objects at rest, or in steady rotation, they play a key role in astrophysics. They are easier to study than non-stationary systems because stationary solutions are governed by elliptic rather than hyperbolic equations. Further, like in any field theory, one expects that large classes of dynamical solutions approach a stationary state in the final stages of their evolution. Last but not least, explicit stationary solutions are easier to come by than dynamical ones. The flagship example is the *Schwarzschild metric*:

$$g = -(1 - \tfrac{2m}{r})dt^2 + \tfrac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2 \,, \tag{1.2.1}$$

$$t \in \mathbb{R}\,, \ r \neq 2m, 0\,. \tag{1.2.2}$$

Here $d\Omega^2$ denotes the metric of the round unit 2-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,.$$

In Section 4.6.5 below we verify that the metric (1.2.2) satisfies the vacuum Einstein equations, see (4.6.38)- (4.6.40), p. 157 (compare Besse [24] for a very different calculation).

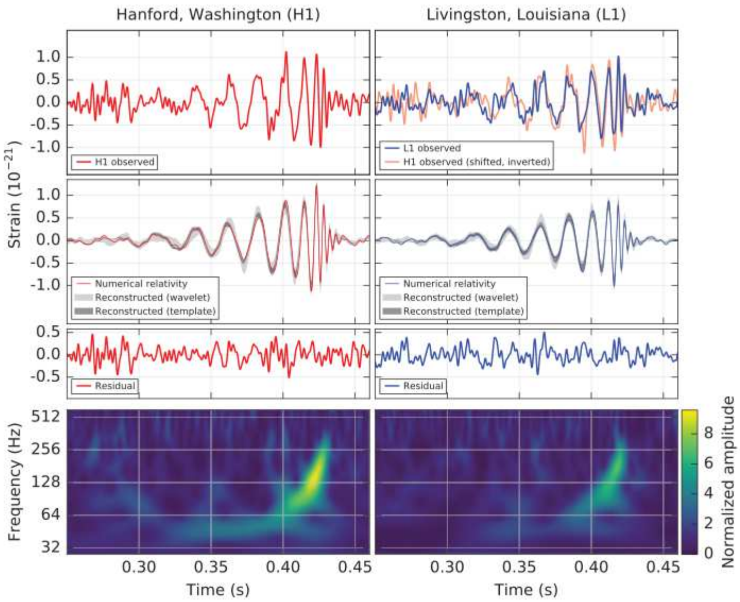

Figure 1.1.4: GW150914 as observed in the Hanford and Livingston detectors, from [2]. The top row is the signal observed, after filtering out the low-frequency and high-frequency noise. In the figure in the top right-corner, the Hanford signal has been inverted when superposing with the Livingston signal because of an opposite orientation of the detector arms. The bottom row shows the evolution of frequency of the signal in time.

Generators of isometries are called *Killing vectors*, we will return to the notion later. A theorem due to Jebsen [164], but usually attributed to Birkhoff [27], shows that:

THEOREM 1.2.1 *Any spherically symmetric vacuum metric has a further* local *Killing vector, say X, orthogonal to the orbits of spherical symmetry. Near any point at which X is* not *null the metric can be* locally *written in the Schwarzschild form (1.2.1), for some mass parameter m.*

INCIDENTALLY: One can find in the literature several results referred to as "Birkhoff theorems", see [253] for an overview. Theorem **??** is a special case of the classification of "warped product spacetimes" in [8], carried-out for various Einstein-matter systems. In fact, in [8] one does not even need the full Einstein equations to be satisfied. Further, existence of isometries is not assumed, instead one considers metrics of a block-diagonal form which would follow in the presence of a suitable group of isometries.

Specializing [8, Theorem 3.2] to the case of vacuum spacetimes with a cosmological constant one has:

THEOREM 1.2.3 *Consider a spacetime*

$$(M = Q \times F, \bar{g} = g + r^2 h)$$

satisfying the vacuum Einstein equations with cosmological constant Λ, where (Q, g)

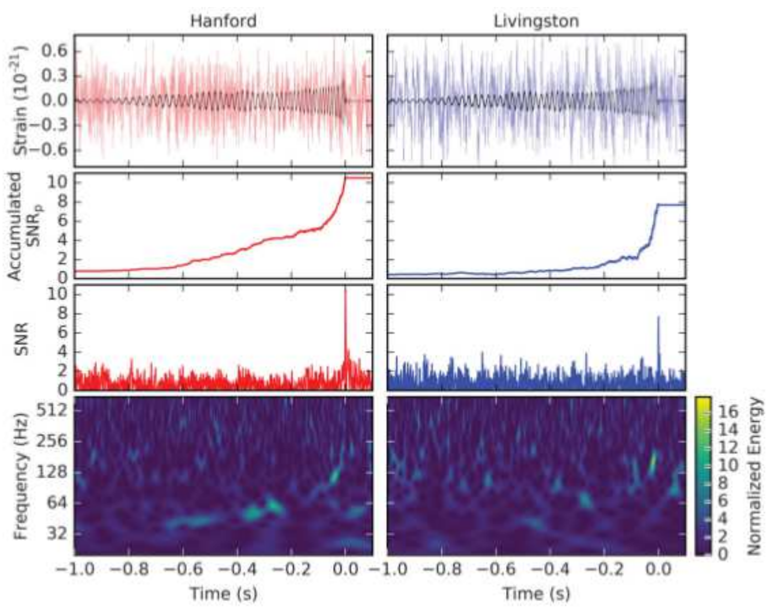

Figure 1.1.5: GW151226 as observed in the Hanford and Livingston detectors, from [1]. The top row is the signal observed, after filtering out the low-frequency and high-frequency noise, superposed with the black curves corresponding to the best-fit general-relativistic template. The second row shows the accumulated-in-time signal-to-noise ratio. The third row shows the Signal-to-noise ratio (SNR) time series produced by time shifting the best-match template waveform and computing the integrated SNR at each point in time. The bottom row show the evolution of frequency of the signal in time.

is a 2-dimensional manifold, (F,h) an $n \geq 2$ dimensional one and r is a function on Q. Then

1. *either $\bar g$ takes the standard Eddington-Finkelstein form*

$$\bar g = -\left(\frac{R^{[h]}}{n(n-1)} - \frac{2m}{r^{n-1}} - \frac{2\Lambda}{n(n+1)} r^2\right) du^2 \pm 2du dr + r^2 h \,,$$

   where $R^{[h]} = \text{const}$ is the scalar curvature of h,

2. *or $\Lambda = 0$, the Ricci tensor of h vanishes, and*

$$\bar g = -dt^2 + dr^2 + (t \pm r)^2 h \,,$$

3. *or r is constant, (Q,g) is maximally symmetric, (F,h) is Einstein, $R^{[h]} = 2r^2\Lambda$, and $R^{[g]} = 4\Lambda/n$.*

When (F,h) is $\mathbb{S}^n$ with the round metric this reduces to the classic Birkhoff theorem. In that case (2) does not apply, and (3) gives the Nariai metrics, cf. Example 4.3.5, p. 145 (see also [252, Section 4]). □

Remark 1.2.4 Note that a locally defined Killing vector does not necessarily extend to a global one. A simple example of this is provided by a flat torus: the collection of

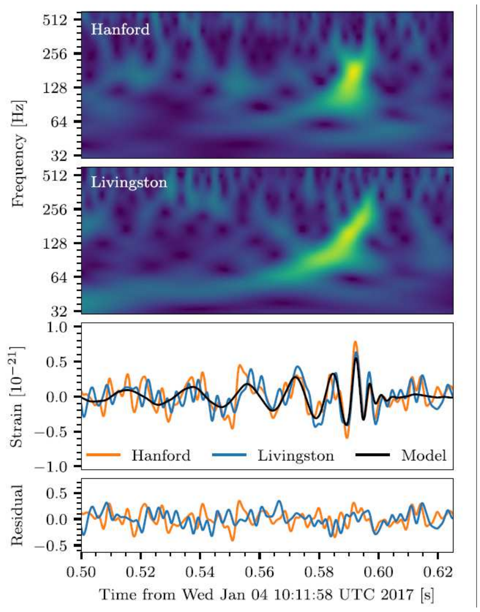

Figure 1.1.6: GW170104 as observed in the Hanford and Livingston detectors, from [3]. The top two rows show the evolution in time of the frequency spectrum of the signal, after filtering out the low- and high-frequency noise. The third row is a superposition of the filtered signals together with the black curve corresponding to the best-fit general-relativistic template. The last row shows residuals from the best fit.

Killing vector fields on sufficiently small balls contains all the generators of rotations and translations, but only the translational Killing vectors extend to globally defined ones. Example 1.4.1, p. 56, is also instructive in this context. □

We conclude that the hypothesis of spherical symmetry implies in vacuum, at least locally, the existence of two further symmetries: translations in t and t–reflections $t \to -t$. More precisely, we obtain *time translations* and *time-reflections* in the region where $1-2m/r > 0$ (a metric with those two properties is called *static*). However, in the region where $r < 2m$ the notation "t" for the coordinate appearing in (1.2.1) is misleading, as t is then a space-coordinate, and r is a time one. So in this region t–translations are actually translations in space.

The above requires some comments and definitions, which will be useful for our further analysis (see also Appendix A.24): First, we need to define the notion of *time orientation*. This is a decision about which timelike vectors are future-pointing, and which ones are past-pointing. In special relativity this is taken for granted: in coordinates where the Minkowski metric η takes the form

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{1.2.3}$$

a timelike vector $X^\mu\partial_\mu$ is said to be future pointing if $X^0 > 0$. But, it should be realized that this is a question of conventions: we could very well agree that

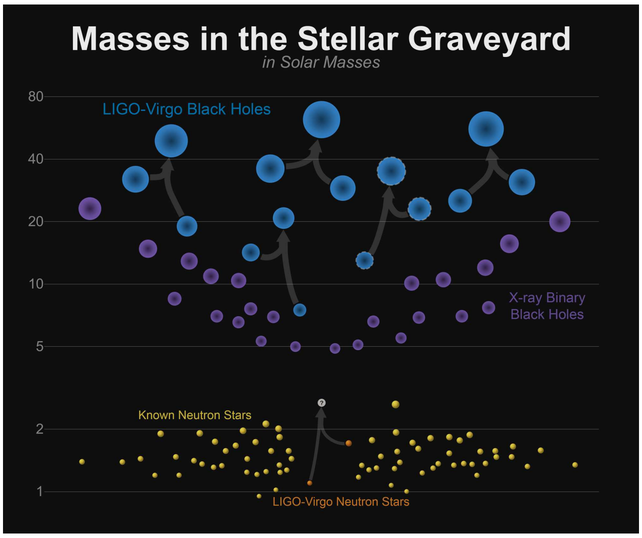

Figure 1.1.7: Neutron stars and black holes with masses up to 100 $M_{☼}$, as known in early 2018, from the Caltech LIGO website.

future-pointing vectors are those with negative X^0. We will shortly meet a situation where such a decision will have to be made.

Next, a function f will be called a *time function* if ∇f is everywhere timelike past pointing. A coordinate, say y^0 will be said to be a *time coordinate* if y^0 is a time function.

So, for example, $f = t$ on Minkowski spacetime is a time function: indeed, in canonical coordinates as in (1.2.3)

$$\nabla t = \eta^{\mu\nu}\partial_\mu t\,\partial_\nu = \eta^{0\nu}\partial_\nu = -\partial_t\,, \tag{1.2.4}$$

and so

$$\eta(\nabla t, \nabla t) = \eta(\partial_t, \partial_t) = -1\,.$$

(The minus sign in (1.2.4) is at the origin of the requirement that ∇f be past pointing, rather than future pointing.)

On the other hand, consider $f = t$ in the Schwarzschild metric: the inverse metric now reads

$$g^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{1-\frac{2m}{r}}\partial_t^2 + (1-\frac{2m}{r})\partial_r^2 + r^{-2}(\partial_\theta^2 + \sin^{-2}\theta\partial_\varphi^2)\,, \tag{1.2.5}$$

and so

$$\nabla t = g^{\mu\nu}\partial_\mu t\,\partial_\nu = g^{0\nu}\partial_\nu = -\frac{1}{1-\frac{2m}{r}}\partial_t\,.$$

The length-squared of ∇t is thus

$$g(\nabla t, \nabla t) = \frac{g(\partial_t, \partial_t)}{(1-\frac{2m}{r})^2} = -\frac{1}{1-\frac{2m}{r}} \begin{cases} < 0, & r > 2m; \\ > 0, & r < 2m. \end{cases}$$

We conclude that t *is a time function in the region* $\{r > 2m\}$ when the usual time orientation is chosen there, but is *not* on the manifold $\{r < 2m\}$.

A similar calculation for ∇r gives

$$\nabla r = (1 - \tfrac{2m}{r})\partial_r \,,$$
$$g(\nabla r, \nabla r) = (1 - \tfrac{2m}{r})^2 g(\partial_r, \partial_r) = (1 - \tfrac{2m}{r}) \begin{cases} > 0, & r > 2m; \\ < 0, & r < 2m. \end{cases}$$

So r is a *time function* in the region $\{r < 2m\}$ *if* the time orientation is chosen so that ∂_r is future pointing. On the other hand, the alternative choice of time-orientation implies that *minus* r is a time function in this region. We return to the implications of this shortly.

1.2.1 The singularity $r = 0$

Unless explicitly indicated otherwise, we will assume

$$m > 0 \,,$$

because $m < 0$ leads to metrics which contain "naked singularities", in the following sense: for $m < 0$, on each spacelike surface $\{t = \text{const}\}$ the set $\{r = 0\}$ can be reached along curves of finite length. But we have (see, *e.g.,* `http://grtensor.phy.queensu.ca/NewDemo`; compare (4.6.22)-(4.6.25), p. 155 below)

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \,, \tag{1.2.6}$$

which shows that the geometry is singular at $r = 0$, whatever $m \in \mathbb{R}^*$.

The advantage of $m > 0$ is the occurrence, as will be seen shortly, of the event horizon $\{r = 2m\}$: the singular set $\{r = 0\}$ is then "hidden" behind an event horizon, which is considered to be less unpleasant than the situation with $m < 0$, where no such horizon occurs.

1.2.2 Eddington-Finkelstein extension

The metric (1.2.1) is singular as $r = 2m$ is approached. It turns out that this singularity is related to a poor choice of coordinates (one talks about "a coordinate singularity"); the simplest way to see it is to replace t by a new coordinate v, which will be chosen to cancel out the singularity in g_{rr}: if we set

$$v = t + f(r) \,,$$

we find $dv = dt + f'dr$, so that

$$\begin{aligned} (1 - \frac{2m}{r})dt^2 &= (1 - \frac{2m}{r})(dv - f'dr)^2 \\ &= (1 - \frac{2m}{r})(dv^2 - 2f'dv\,dr + (f')^2dr^2) \,. \end{aligned}$$

Substituting in (1.2.1), the offending g_{rr} terms will go away if we choose f to satisfy

$$(1 - \frac{2m}{r})(f')^2 = \frac{1}{1 - \frac{2m}{r}} \,.$$

There are two possibilities for the sign; we choose

$$f' = \frac{1}{1-\frac{2m}{r}} = \frac{r}{r-2m} = \frac{r-2m+2m}{r-2m} = 1 + \frac{2m}{r-2m}\,, \tag{1.2.7}$$

leading to

$$v = t + r + 2m\ln\left(\frac{r-2m}{2m}\right). \tag{1.2.8}$$

The alternative choice amounts to introducing another coordinate

$$u = t - f(r)\,, \tag{1.2.9}$$

with f still as in (1.2.7).

The choice (1.2.8) brings g to the form

$$g = -\left(1-\frac{2m}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2\,, \tag{1.2.10}$$

and note that the choice (1.2.9) would lead to a non-diagonal term $-2du\,dr$ instead in the metric above. Now, all coefficients of g in the new coordinate system are smooth. Further,

$$\det g = -r^4\sin^2\theta\,,$$

which is non-zero for $r > 0$ except at the north and south pole, where we have the usual spherical-coordinates singularity. Since g has signature $(-,+,+,+)$ for $r > 2m$, the signature cannot change across $r = 2m$, as for this the determinant would have had to vanish there. We conclude that g is a well defined smooth Lorentzian metric on the set

$$\{v\in\mathbb{R}\,, r\in(0,\infty)\}\times S^2\,. \tag{1.2.11}$$

More precisely, (1.2.10)-(1.2.11) defines an analytic extension of the original spacetime (1.2.1).

The coordinates (v, r, θ, φ) are called *"retarded Eddington-Finkelstein coordinates"*.

We claim:

THEOREM 1.2.5 *The region $\{r \le 2m\}$ for the metric (1.2.10) is a* black hole region*, in the sense that*

$$\textit{observers, or signals, can enter this region, but can never leave it.} \tag{1.2.12}$$

PROOF: We have already seen that either r or *minus* r is a time function on the region $\{r < 2m\}$. Now, recall that observers in general relativity always move on *future directed timelike curves*, that is, curves with timelike future directed tangent vector. But time functions are strictly monotonous along future directed causal curves: indeed, let $\gamma(s)$ be such a curve, and let f be a time function, then

$$\frac{d(f\circ\gamma)}{ds} = \dot\gamma^\mu\partial_\mu f = \dot\gamma^\mu g_{\mu\nu}g^{\sigma\nu}\partial_\sigma f = g_{\mu\nu}\dot\gamma^\mu\nabla^\nu f\,.$$

Since $\dot\gamma$ is causal future directed and ∇f is timelike past directed, their scalar product is positive, as desired.

It follows that, along a future directed causal curve, either r or $-r$ is strictly increasing in the region $\{r < 2m\}$.

Suppose that there exists at least one future directed causal curve γ_0 which enters from $r > 2m$ to $r < 2m$. Then r must have been decreasing somewhere along γ_0 in the region $\{r < 2m\}$. This implies that *the time orientation has to be chosen so that $-r$ is a time function.* But then r is decreasing along every causal future directed γ. So no such curve passing through $\{r < 2m\}$ can cross $\{r = 2m\}$ again, when followed to the future.

To finish the proof, it remains to exhibit one future directed causal γ_0 which enters $\{r < 2m\}$ from the region $\{r > 2m\}$. For this, consider the radial curves

$$\gamma_0(s) = (v(s), r(s), \theta(s), \varphi(s)) = (v_0, -s, \theta_0, \varphi_0)\,.$$

Then $\dot\gamma_0 = -\partial_r$, hence

$$g(\dot\gamma_0, \dot\gamma_0) = g_{rr} = 0$$

in the (v, r, θ, φ) coordinates, see (1.2.10). We see that γ_0 lies in the region $\{r > 2m\}$ for $s < -2m$, is null (hence causal), and crosses $\{r = 2m\}$ at $s = -2m$. Finally, we have

$$t(s) = v(s) - f(r(s)) = v_0 - f(r(s))\,,$$

hence

$$\frac{dt(s)}{ds} = -f'(r(s))\frac{dr}{ds} = f'(r(s)) > 0 \quad \text{for } r(s) > 2m,$$

which shows that t is increasing along γ_0 in the region $\{r > 2m\}$, hence γ_0 is future directed there, which concludes the proof. □

Incidentally: An alternative shorter, but perhaps less transparent, argument proceeds as follows: Let $\gamma(s) = (v(s), r(s), \theta(s), \varphi(s))$ be a future directed timelike curve; for the metric (1.2.10) the condition $g(\dot\gamma, \dot\gamma) < 0$ reads

$$-(1 - \frac{2m}{r})\dot v^2 + 2\dot v\dot r + r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) < 0\,.$$

This implies

$$\dot v\Big(- (1 - \frac{2m}{r})\dot v + 2\dot r\Big) < 0\,.$$

It follows that $\dot v$ does not change sign on a timelike curve. As already pointed out, the standard choice of time orientation in the exterior region corresponds to $\dot v > 0$ on future directed curves, so $\dot v$ has to be positive everywhere, which leads to

$$-(1 - \frac{2m}{r})\dot v + 2\dot r < 0\,.$$

For $r \le 2m$ the first term is non-negative, which enforces $\dot r < 0$ on all future directed timelike curves in that region. Thus, r is a strictly decreasing function along such curves, which implies that future directed timelike curves can cross the hypersurface $\{r = 2m\}$ only if coming from the region $\{r > 2m\}$. The same conclusion applies for future directed causal curves: it suffices to approximate a causal curve by a sequence of future directed timelike ones. □

The last theorem motivates the name *black hole event horizon* for $\{r = 2m, v \in \mathbb{R}\} \times S^2$.

INCIDENTALLY: The analogous construction using the coordinate u instead of v leads to a *white hole* spacetime, with $\{r = 2m\}$ being a *white hole event horizon*. The latter can only be crossed by those future directed causal curves which originate in the region $\{r < 2m\}$. In either case, $\{r = 2m\}$ is a causal membrane which prevents future directed causal curves to go *back and forth*. This will become clearer in Section 1.2.3. □

From (1.2.10) one easily finds the inverse metric:

$$g^{\mu\nu}\partial_\mu\partial_\nu = 2\partial_v\partial_r + (1 - \frac{2m}{r})\partial_r^2 + r^{-2}\partial_\theta^2 + r^{-2}\sin^{-2}\theta\partial_\varphi^2 \ . \tag{1.2.13}$$

In particular

$$0 = g^{vv} = g(\nabla v, \nabla v)\,,$$

which implies that the integral curves of

$$\nabla v = \partial_r$$

are null, affinely parameterised geodesics: Indeed, let $X = \nabla v$, then

$$X^\alpha \nabla_\alpha X^\beta = \nabla^\alpha v \nabla_\alpha \nabla^\beta v = \nabla^\alpha v \nabla^\beta \nabla_\alpha v = \frac{1}{2}\nabla^\beta(\nabla^\alpha v \nabla_\alpha v) = 0\,. \tag{1.2.14}$$

So if γ is an integral curve of X (by definition, this means that

$$\dot\gamma^\mu = X^\mu)\,, \tag{1.2.15}$$

we obtain the geodesic equation:

$$X^\alpha \nabla_\alpha X^\beta = \dot\gamma^\alpha \nabla_\alpha \dot\gamma^\beta = 0\,. \tag{1.2.16}$$

We also have

$$g(\nabla r, \nabla r) = g^{rr} = \left(1 - \frac{2m}{r}\right)\,, \tag{1.2.17}$$

and since this vanishes at $r = 2m$ we say that the hypersurface $r = 2m$ is *null*. It is reached by all the radial null geodesics $v = \text{const}$, $\theta = \text{const}'$, $\varphi = \text{const}''$, in finite affine time.

The calculation leading to (1.2.16) generalizes to functions f such that ∇f satisfies an equation of the form

$$g(\nabla f, \nabla f) = \psi(f)\,, \tag{1.2.18}$$

for some function ψ; note that $f = r$ satisfies this, in view of (1.2.17); see Proposition A.13.2. Thus the integral curves of ∇r are geodesics as well. Now, in the (v, r, θ, φ) coordinates one finds from (1.2.13)

$$\nabla r = \partial_v + (1 - \frac{2m}{r})\partial_r\,,$$

which equals ∂_v at $r = 2m$. So the curves $(v = s, r = 2m, \theta = \theta_0, \varphi = \varphi_0)$ are null geodesics. They are called *generators* of the event horizon.

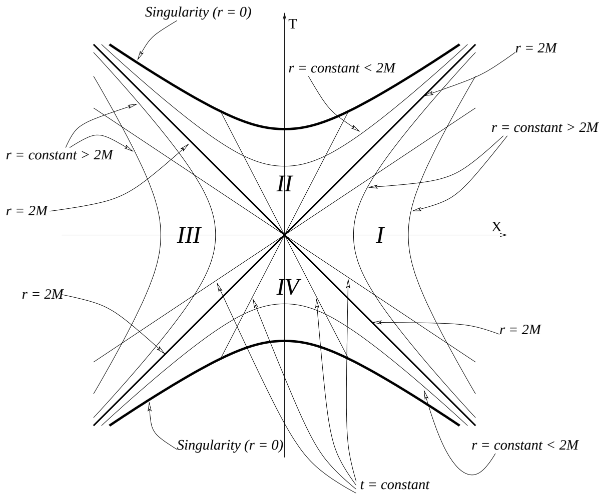

Figure 1.2.1: The Kruskal-Szekeres extension of the Schwarzschild solution.

1.2.3 The Kruskal-Szekeres extension

The transition from (1.2.1) to (1.2.10) is not the end of the story, as further extensions are possible, which will be clear from the calculations that we will do shortly. For the metric (1.2.1) a maximal analytic extension has been found independently by Kruskal [176], Szekeres [261], and Fronsdal [124]; for some obscure reason Fronsdal is almost never mentioned in this context. This extension is visualised[5] in Figure 1.2.1. The region I there corresponds to the spacetime (1.2.1), while the extension just constructed corresponds to the regions I and II.

The general construction for spherically symmetric metrics proceeds as follows: Let us write the metric in the form

$$g = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \tag{1.2.19}$$

where V^2 is a smooth function which depends only upon r and which we allow to be negative. We introduce another coordinate u, defined by changing a sign in (1.2.7)

$$u = t - f(r) \,, \quad f' = \frac{1}{V^2} \,, \tag{1.2.20}$$

leading to

$$u = t - r - 2m \ln\left(\frac{r - 2m}{2m}\right) \,.$$

We could now replace (t, r) by (u, r), obtaining an extension of the exterior region I of Figure 1.2.1 into the "white hole" region IV. We leave that extension as an exercise for the reader, and we pass to the complete extension, which proceeds in two steps. First, we replace (t, r) by (u, v). We note that

$$V\, du = V\, dt - \frac{1}{V} dr \,, \quad V\, dv = V\, dt + \frac{1}{V} dr \,,$$

[5] I am grateful to J.-P. Nicolas for allowing me to use his electronic figures from [219].

which gives

$$V\,dt = \frac{V}{2}(du+dv)\,, \quad \frac{1}{V}dr = \frac{V}{2}(dv-du)\,.$$

Inserting this into (1.2.1) brings g to the form

$$\begin{aligned} g &= -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega^2 \\ &= \frac{V^2}{4}\Big(-(du+dv)^2 + (du-dv)^2\Big) + r^2d\Omega^2 \\ &= -V^2du\,dv + r^2d\Omega^2\,. \end{aligned} \tag{1.2.21}$$

The metric so obtained is still degenerate at $\{V=0\}$. The desingularisation is now obtained by setting

$$\hat{u} = -\exp(-cu)\,, \quad \hat{v} = \exp(cv)\,, \tag{1.2.22}$$

with an appropriately chosen c: since

$$d\hat{u} = c\exp(-cu)\,du\,, \quad d\hat{v} = c\exp(cv)dv\,,$$

we obtain

$$\begin{aligned} V^2du\,dv &= \frac{V^2}{c^2}\exp(-c(-u+v))d\hat{u}\,d\hat{v} \\ &= \frac{V^2}{c^2}\exp(-2cf(r))d\hat{u}\,d\hat{v}\,. \end{aligned}$$

In the Schwarzschild case this reads

$$\begin{aligned} \frac{V^2}{c^2}\exp(-2cf(r)) &= \frac{r-2m}{c^2r}\exp\left(-2c\left(r+2m\ln\left(\frac{r-2m}{2m}\right)\right)\right) \\ &= \frac{\exp(-2cr)}{c^2r}(r-2m)\exp\left(-4mc\ln\left(\frac{r-2m}{2m}\right)\right)\,, \end{aligned}$$

and with the choice

$$4mc = 1$$

the term $r-2m$ cancels out, leading to a factor in front of $d\hat{u}\,d\hat{v}$ which has no zeros for $r \neq 0$ near. Thus, the desired coordinate transformation is

$$\hat{u} = -\exp(-cu) = -\exp(\tfrac{r-t}{4m})\sqrt{\tfrac{r-2m}{2m}}\,, \tag{1.2.23}$$

$$\hat{v} = \exp(cv) = \exp(\tfrac{r+t}{4m})\sqrt{\tfrac{r-2m}{2m}}\,, \tag{1.2.24}$$

with g taking the form

$$\begin{aligned} g &= -V^2du\,dv + r^2d\Omega^2 \\ &= -\frac{32m^3\exp(-\frac{r}{2m})}{r}d\hat{u}\,d\hat{v} + r^2d\Omega^2\,. \end{aligned} \tag{1.2.25}$$

Here r should be viewed as a function of $\hat{u}$ and $\hat{v}$ defined implicitly by the equation

$$-\hat{u}\hat{v} = \underbrace{\exp\left(\frac{r}{2m}\right)\frac{(r-2m)}{2m}}_{=:G(r)}\,. \tag{1.2.26}$$

Indeed, we have

$$\Big(\exp(\frac{r}{2m})(r-2m)\Big)' = \frac{r}{2m}\exp(\frac{r}{2m}) > 0\,,$$

which shows that the function G defined at the right-hand side of (1.2.26) is a smooth strictly increasing function of $r > 0$. We have $G(0) = -1$, and G tends to infinity as r does, so G defines a bijection of $(0,\infty)$ with $(-1,\infty)$. The implicit function theorem guarantees smoothness of the inverse G^{-1}, and hence the existence of a smooth function $r = G^{-1}(-\hat u\hat v)$ solving (1.2.26) on the set $\hat u\hat v \in (-\infty, 1)$.

Note that so far we had $r > 2m$, but there are *a priori* no reasons for the function $r(u,v)$ defined above to satisfy this constraint. In fact, we already know from our experience with the (v, r, θ, φ) coordinate system that a restriction $r > 2m$ would lead to a spacetime with poor global properties.

We have $\det g = -(32m^3)^2\exp(-\frac{r}{m})r^2\sin^2\theta$, with all coefficients of g smooth, which shows that (1.2.25) defines a smooth Lorentzian metric on the set

$$\{\hat u,\ \hat v \in \mathbb{R} \text{ such that } r > 0\}\,. \tag{1.2.27}$$

This is the *Kruszkal-Szekeres* extension of the original spacetime (1.2.1). Figure 1.2.1 gives a representation of the extended spacetime in coordinates

$$X = (\hat v - \hat u)/2\,, \quad T = (\hat v + \hat u)/2\,. \tag{1.2.28}$$

Since (1.2.6) shows that the so-called *Kretschmann scalar* $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r^{-6} when r approaches zero, we conclude that the metric cannot be extended across the set $r = 0$, at least in the class of C^2 metrics.

Let us discuss some features of Figure 1.2.1:

1. The singular set $r = 0$ corresponds to the spacelike hyperboloids
$$(T^2 - X^2)|_{r=0} = \hat u\hat v|_{r=0} = 1 > 0\,.$$

2. More generally, the sets $r = \text{const}$ are hyperboloids $X^2 - T^2 = \text{const}'$, which are timelike in the regions I and III (since $X^2 - T^2 < 0$ there), and which are spacelike in the regions II and IV.

3. The vector field ∇T satisfies
$$g(\nabla T, \nabla T) = g^\sharp(dT, dT) = \frac{1}{4}g^\sharp(d\hat u + d\hat v, d\hat u + d\hat v) = \frac{1}{2}g^\sharp(d\hat u, d\hat v) < 0\,,$$
which shows that T is a time coordinate. Similarly X is a space-coordinate, so that Figure 1.2.1 respects our implicit convention of representing time along the vertical axis and space along the horizontal one.

4. The map
$$(\hat u, \hat v) \to (-\hat u, -\hat v)$$
is clearly an isometry, so that the region I is isometric to region III, and region II is isometric to region IV. In particular the extended manifold has two asymptotically flat regions, the original region I, and region III which is an identical copy I.

5. The hypersurface $t = 0$ from the region I corresponds to $\hat{v} = -\hat{u} > 0$, equivalently it is the subset $X > 0$ of the hypersurface $T = 0$. This can be smoothly continued to negative X, which corresponds to a second copy of this hypersurface. The resulting geometry is often referred to as the *Einstein-Rosen bridge.* It is instructive to do the continuation directly using the Riemannian metric γ induced by g on $t = 0$:
$$\gamma = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 \,, \quad r > 2m \,.$$
A convenient coordinate ρ is given by
$$\rho = \sqrt{r^2 - 4m^2} \iff r = \sqrt{\rho^2 + 4m^2} \,.$$
This brings γ to the form
$$\gamma = \Big(1 + \frac{2m}{\sqrt{\rho^2 + 4m^2}}\Big) d\rho^2 + (\rho^2 + 4m^2) d\Omega^2 \,, \tag{1.2.29}$$
which can be smoothly continued from the original range $\rho > 0$ to $\rho \in \mathbb{R}$. Equation (1.2.29) further exhibits explicitly asymptotic flatness of both asymptotic regions $\rho \to \infty$ and $\rho \to -\infty$. Indeed,
$$\gamma \sim d\rho^2 + \rho^2 d\Omega^2$$
to leading order, for large $|\rho|$, which is the flat metric in radial coordinates with radius $|\rho|$.

6. In order to understand how the Eddington-Finkelstein extension using the v coordinate fits into Figure 1.2.1, we need to express $\hat{u}$ in terms of v and r. For this we have
$$u = t - f(r) = v - 2f(r) = v - 2r - 4m \ln\big(\frac{r}{2m} - 1\big) \,,$$
hence
$$\hat{u} = -e^{-\frac{u}{4m}} = -e^{-\frac{v-2r}{4m}}\big(\frac{r}{2m} - 1\big) \,, \quad \hat{v} = e^{\frac{v}{4m}} \,.$$
So $\hat{v}$ remains positive but $\hat{u}$ is allowed to become negative as r crosses $r = 2m$ from above. This corresponds to the region above the diagonal $T = -X$ in the coordinates (X, T) of Figure 1.2.1.

   A similar calculation shows that the Eddington-Finkelstein extension using the coordinate u corresponds to the region $\hat{u} < 0$ within the Kruszkal-Szekeres extension, which is the region below the diagonal $T = X$ in the coordinates of Figure 1.2.1.

7. Vector fields generating isometries are called *Killing vector fields.* Since time-translations are isometries in our case, $K = \partial_t$ is a Killing vector field. In the Kruskal-Szekeres coordinate system the Killing vector field $K = \partial_t$ takes the form
$$\begin{aligned} K &= \partial_t = \frac{\partial \hat{u}}{\partial t}\partial_{\hat{u}} + \frac{\partial \hat{v}}{\partial t}\partial_{\hat{v}} \\ &= \frac{1}{4m}\left(-\hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}}\right) \,. \end{aligned} \tag{1.2.30}$$

More precisely, the Killing vector field ∂_t defined on the original Schwarzschild region extends to a Killing vector field X defined throughout the Kruskal-Szekeres manifold by the second line of (1.2.30).

We note that K is tangent to the level sets of $\hat{u}$ or $\hat{v}$ at $\hat{u}\hat{v} = 0$, and therefore is null there. Moreover, it vanishes at the sphere $\hat{u} = \hat{v} = 0$, which is called *the bifurcation surface of a bifurcate Killing horizon.* The justification of this last terminology should be clear from Figure 1.2.1.

A hypersurface $\mathscr{H}$ is called *null* if the pull-back of the spacetime metric to $\mathscr{H}$ is degenerate, see Appendix A.23, p. 316. Quite generally, an embedded null hypersurface to which a Killing vector is tangent, and null there, is called a *Killing horizon.*[6] Therefore the union $\{\hat{u}\hat{v} = 0\}$ of the black hole horizon $\{\hat{u} = 0\}$ and the white hole event horizon $\{\hat{v} = 0\}$ can be written as the union of four Killing horizons and of their bifurcation surface.

The bifurcate horizon structure, as well as the formula (1.2.30), are rather reminiscent of what happens when considering the Killing vector $t\partial_x + x\partial_t$ in Minkowski spacetime; this is left as an exercice to the reader.

The Kruskal-Szekeres extension is *inextendible* within the class of C^2-extensions (compare Theorem 1.2.10 below), which can be proved as follows: First, (1.2.6) shows that the *Kretschmann scalar* $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r approaches zero. As already pointed out, this implies that no C^2 extension of the metric is possible across the set $\{r = 0\}$. Next, an analysis of the geodesics of the Kruskal-Szekeres metric shows that all (maximally extended) geodesics which *do not* approach $\{r = 0\}$ are complete. This, Theorem 1.4.2 and Proposition 1.4.3 below implies inextendibility. Together with Corollary 1.4.7 we thus obtain:

THEOREM 1.2.8 *The Kruskal-Szekeres spacetime is the unique extension, within the class of simply connected analytic extensions of the Schwarzschild region $r > 2m$, with the property that all maximally extended causal geodesics on which $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded are complete.* □

REMARK 1.2.9 Nevertheless, it should be realised that the exterior Schwarzschild spacetime (1.2.1) admits many non-isometric vacuum extensions, *even in the class of maximal, analytic, simply connected ones*: indeed, let S be any two-dimensional closed submanifold entirely included in, say, the black-hole region of the Kruskal-Szekeres manifold $(\mathscr{M}, g)$, such that $\mathscr{M} \setminus S$ is *not* simply connected. (A natural example is obtained by removing the "bifurcation sphere" $\{\hat{u} = \hat{v} = 0\}$.) Then, for any such S the universal covering manifold $(\mathscr{M}_S, \hat{g})$ of $(\mathscr{M} \setminus S, g|_{\mathscr{M}\setminus S})$ has the claimed properties. While maximal, these extensions will contain inextendible geodesics on which the geometry is bounded, consistently with Theorem 1.2.8. We return to such issues in Section 1.4 below.

Yet another particulary interesting extension of the region $\{r > 2m\}$ is provided by the "$\mathbb{RP}^3$ geon" of [118], see Example 1.4.1. □

[6] More precisely, let X be a Killing vector field. A *Killing horizon* is a connected component of the set $\{g(X,X) = 0\,,\ X \neq 0\}$ which forms an embedded hypersurface.

A beautiful theorem of Sbierski [249] asserts that:

THEOREM 1.2.10 *The Kruskal-Szekeres spacetime is inextendible within the class of* Lorentzian spacetimes with continuous metrics.

1.2.4 Other coordinate systems, higher dimensions

A convenient coordinate system for the Schwarzschild metric is given by the so-called *isotropic coordinates*: introducing a new radial coordinate $\tilde{r}$, implicitly defined by the formula

$$r = \tilde{r}\Big(1 + \frac{m}{2\tilde{r}}\Big)^2\,, \tag{1.2.31}$$

with a little work one obtains

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 \left(\sum_{1=1}^{3} (dx^i)^2\right) - \left(\frac{1 - m/2|x|}{1 + m/2|x|}\right)^2 dt^2\,, \tag{1.2.32}$$

where x^i are coordinates on $\mathbb{R}^3$ with $|x| = \tilde{r}$. Those coordinates show explicitly that the space-part of the metric is conformally flat (as follows from spherical symmetry).

The Schwarzschild spacetime has the curious property of possessing *flat* spacelike hypersurfaces. They appear miraculously when introducing the *Painlevé–Gullstrand coordinates* [145, 184, 227]: Starting from the standard coordinate system of (1.2.1) one introduces a new time τ via the equation

$$t = \tau - 2r\sqrt{\frac{2m}{r}} + 4m\ \operatorname{arctanh}\left(\sqrt{\frac{2m}{r}}\right)\,, \tag{1.2.33}$$

so that

$$dt = d\tau - \frac{\sqrt{2m/r}}{1 - 2m/r}\, dr\,.$$

This leads to

$$g = -\left[1 - \frac{2m}{r}\right] d\tau^2 + 2\sqrt{\frac{2m}{r}}\, dr\, d\tau + dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right]\,,$$

or, passing from spherical to standard coordinates,

$$g \;=\; -\left[1 - \frac{2m}{r}\right] d\tau^2 + 2\sqrt{\frac{2m}{r}}\, dr\, d\tau + dx^2 + dy^2 + dz^2\,. \tag{1.2.34}$$

(Note that each such slice has zero ADM mass.)

A useful tool for the PDE analysis of spacetimes is provided by *wave coordinates*. In spherical coordinates associated to wave coordinates $(t, \hat{x}, \hat{y}, \hat{z})$, with radius function $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2}$, the Schwarzschild metric takes the form [188, 259]

$$g = -\frac{\hat{r} - m}{\hat{r} + m} dt^2 + \frac{\hat{r} + m}{\hat{r} - m} d\hat{r}^2 + (\hat{r} + m)^2 d\Omega^2\,. \tag{1.2.35}$$

This is clearly obtained by replacing r with $\hat{r} = r - m$ in (1.2.1).

INCIDENTALLY: In order to verify the harmonic character of the coordinates associated with (1.2.35), consider a general spherically symmetric static metric of the form

$$\begin{aligned} g &= -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + e^{2\gamma}r^2 d\Omega^2 \\ &= -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + e^{2\gamma}(\delta_{ij}dx^i dx^j - dr^2) \\ &= -e^{2\alpha}dt^2 + \Big(e^{2\gamma}\delta_{ij} + (e^{2\beta} - e^{2\gamma})\frac{x^i x^j}{r^2}\Big)dx^i dx^j \,, \end{aligned} \tag{1.2.36}$$

where α, β and γ depend only upon r. We need to calculate

$$\Box_g x^\alpha = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu x^\alpha) = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu\alpha})\,.$$

Clearly $g^{0i} = 0$, which makes the calculation for $x^0 = t$ straightforward:

$$\Box_g t = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}g^{\mu 0}) = \frac{1}{\sqrt{|\det g|}}\partial_t(\sqrt{|\det g|}g^{00}) = 0\,,$$

as nothing depends upon t. For $\Box_g x^i$ we have to calculate $\sqrt{|\det g|}$ and $g^{\mu\nu}$. For the latter, it is clear that $g^{00} = -e^{-2\alpha}$, while by symmetry considerations we must have

$$g^{ij} = e^{-2\gamma}\Big(\delta^{ij} + \chi\frac{x^i x^j}{r^2}\Big)\,,$$

for a function χ to be determined. The equation

$$\begin{aligned} \delta_i^j &= g^{j\mu}g_{\mu i} = g^{jk}g_{ki} = e^{-2\gamma}\Big(\delta^{jk} + \chi\frac{x^j x^k}{r^2}\Big)\Big(e^{2\gamma}\delta_{ki} + (e^{2\beta} - e^{2\gamma})\frac{x^k x^i}{r^2}\Big) \\ &= \delta_i^j + e^{-2\gamma}\Big(\chi e^{2\gamma} + e^{2\beta} - e^{2\gamma} + \chi(e^{2\beta} - e^{2\gamma})\Big)\frac{x^i x^j}{r^2} \\ &= \delta_i^j + e^{-2\gamma}\Big(e^{2\beta} - e^{2\gamma} + \chi e^{2\beta}\Big)\frac{x^i x^j}{r^2} \end{aligned}$$

gives $\chi = e^{2(\gamma-\beta)} - 1$, and finally

$$g^{ij} = e^{-2\gamma}\delta^{ij} + (e^{-2\beta} - e^{-2\gamma})\frac{x^i x^j}{r^2}\,.$$

Next, $\sqrt{|\det g|}$ is best calculated in a coordinate system in which the vector (x, y, z) is aligned along the x axis, $(x, y, z) = (r, 0, 0)$. Then (1.2.36) reads, in spacetime dimension $n + 1$,

$$g = \begin{pmatrix} -e^{2\alpha} & 0 & 0 & \cdots & 0 \\ 0 & e^{2\beta} & 0 & \cdots & 0 \\ 0 & 0 & e^{2\gamma} & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & e^{2\gamma} \end{pmatrix}$$

which implies

$$\det g = -e^{2(\alpha+\beta)+2(n-1)\gamma}\,,$$

still at $(x, y, z) = (r, 0, 0)$. Spherical symmetry implies that this equality holds everywhere.

In order to continue, it is convenient to set

$$\phi = e^{\alpha+\beta+(n-3)\gamma} \qquad \psi = e^{\alpha+\beta+(n-1)\gamma}(e^{-2\beta} - e^{-2\gamma})\,.$$

We then have

$$\begin{aligned}
\sqrt{|\det g|}\Box_g x^i &= \partial_\mu(\sqrt{|\det g|}g^{\mu i}) = \partial_j(\sqrt{|\det g|}g^{ji}) \\
&= \partial_j\Big(\underbrace{e^{\alpha+\beta+(n-3)\gamma}}_{\phi}\delta^{ij} + \underbrace{e^{\alpha+\beta+(n-1)\gamma}(e^{-2\beta}-e^{-2\gamma})}_{\psi}\frac{x^ix^j}{r^2})\Big) \\
&= (\phi'+\psi')\frac{x^i}{r} + \psi\partial_j\Big(\frac{x^ix^j}{r^2}\Big) = \Big(\phi'+\psi'+\frac{(n-1)}{r}\psi\Big)\frac{x^i}{r}\,.
\end{aligned} \tag{1.2.37}$$

For the metric (1.2.35) we have

$$e^{2\alpha} = \frac{\hat r - m}{\hat r + m}\,,\quad \beta = -\alpha\,,\quad e^{2\gamma}\hat r^2 = (\hat r + m)^2\,,$$

so that

$$\phi = 1\,,\quad \psi = e^{2\gamma}\times e^{2\alpha} - 1 = \frac{(\hat r+m)^2}{\hat r^2}\times\frac{\hat r - m}{\hat r + m} - 1 = -\frac{m^2}{\hat r^2}\,,$$

and if $n=3$ we obtain $\Box_g x^\mu = 0$, as desired.

More generally, consider the Schwarzschild metric in any dimension $n \ge 3$,

$$g_m = -\left(1 - \frac{2m}{r^{n-2}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r^{n-2}}} + r^2 d\Omega^2\,, \tag{1.2.38}$$

where, as usual, $d\Omega^2$ is the round unit metric on S^{n-1}. In order to avoid confusion we keep the symbol r for the coordinate appearing in (1.2.38), and rewrite (1.2.36) as

$$g \;=\; -e^{2\alpha}dt^2 + e^{2\beta}d\hat r^2 + e^{2\gamma}\hat r^2 d\Omega^2\,, \tag{1.2.39}$$

It follows from (1.2.37) that the harmonicity condition reads

$$0 = \frac{d(\phi+\psi)}{d\hat r} + \frac{(n-1)}{\hat r}\psi = \frac{d(\phi+\psi)}{d\hat r} + \frac{(n-1)}{\hat r}(\psi+\phi) - \frac{(n-1)}{\hat r}\phi\,. \tag{1.2.40}$$

Equivalently,

$$\frac{d[\hat r^{n-1}(\phi+\psi)]}{d\hat r} = (n-1)\hat r^{n-2}\phi\,. \tag{1.2.41}$$

Transforming r to $\hat r$ in (1.2.68) and comparing with (1.2.39) we find

$$e^\alpha = \sqrt{1-\frac{2m}{r^{n-2}}}\,,\quad e^\beta = e^{-\alpha}\frac{dr}{d\hat r}\,,\quad e^\gamma = \frac{r}{\hat r}\,.$$

Note that $\phi+\psi = e^{\alpha-\beta+(n-1)\gamma}$; chasing through the definitions one obtains $\phi = \frac{dr}{d\hat r}\left(\frac{r}{\hat r}\right)^{n-3}$, leading eventually to the following form of (1.2.41)

$$\frac{d}{dr}\Big[r^{n-1}\Big(1-\frac{2m}{r^{n-2}}\Big)\frac{d\hat r}{dr}\Big] = (n-1)r^{n-3}\hat r\,.$$

Introducing $x = 1/r$, one obtains an equation with a Fuchsian singularity at $x=0$:

$$\frac{d}{dx}\Big[x^{3-n}\Big(1-2mx^{n-2}\Big)\frac{d\hat r}{dx}\Big] = (n-1)x^{1-n}\hat r\,.$$

The characteristic exponents are -1 and $n-1$ so that, after matching a few leading coefficients, the standard theory of such equations provides solutions with the behavior

$$\hat{r} = r - \frac{m}{(n-2)r^{n-3}} + \begin{cases} \frac{m^2}{4} r^{-3} \ln r + O(r^{-5} \ln r), & n = 4; \\ O(r^{5-2n}), & n \geq 5. \end{cases} .$$

Somewhat surprisingly, we find logarithms of r in an asymptotic expansion of $\hat{r}$ in dimension $n = 4$. However, for $n \geq 5$ there is a complete expansion of $\hat{r}$ in terms of inverse powers of r, without any logarithmic terms in those dimensions. □

As already hinted to in (1.2.38), higher dimensional counterparts of metrics (1.2.1) have been found by Tangherlini [263]. In spacetime dimension $n+1$, the metrics take the form (1.2.1) with

$$V^2 = 1 - \frac{2m}{r^{n-2}} , \tag{1.2.42}$$

and with $d\Omega^2$ — the unit round metric on S^{n-1}. The parameter m is the *Arnowitt-Deser-Misner mass* in spacetime dimension four, and is proportional to that mass in higher dimensions. Assuming again $m > 0$, a maximal analytic extension can be constructed by a simple modification of the calculations above, leading to a spacetime with global structure identical to that of Figure 1.2.7 except for the replacement $2M \to (2M)^{1/(n-2)}$ there.

Remark 1.2.12 For further reference we present a general construction of Walker [274]. We summarise the calculations already done: the starting point is a metric of the form

$$g = -F dt^2 + F^{-1} dr^2 + \underbrace{h_{AB} dx^A dx^B}_{=:h} , \tag{1.2.43}$$

with $F = F(r)$, where $h := h_{AB}(t, r, x^C) dx^A dx^B$ is a family of Riemannian metrics on an $(n-2)$-dimensional manifold which possibly depend on t and r. It is convenient to write F for V^2, as the sign of F did not play any role; similarly the metric h was irrelevant for the calculations we did above. We assume that F is defined for r in a neighborhood of $r = r_0$, at which F vanishes, with a simple zero there. Equivalently,

$$F(r_0) = 0 , \quad F'(r_0) \neq 0 .$$

Defining

$$u = t - f(r) , \quad v = t + f(r) , \quad f' = \frac{1}{F} , \tag{1.2.44}$$

$$\hat{u} = -\exp(-cu) , \quad \hat{v} = \exp(cv) , \tag{1.2.45}$$

one is led to the following form of the metric

$$g = -\frac{F}{c^2} \exp(-2cf(r)) d\hat{u}\, d\hat{v} + h . \tag{1.2.46}$$

Since F has a simple zero, it factorizes as

$$F(r) = (r - r_0) H(r) , \quad H(r_0) = F'(r_0) ,$$

for a function H which has no zeros in a neighborhood of r_0. This follows immediately from the formula

$$F(r)-F(r_0) = \int_0^1 \frac{dF(t(r-r_0)+r_0)}{dt}\,dt = (r-r_0)\int_0^1 F'(t(r-r_0)+r_0)\,dt\,. \quad (1.2.47)$$

Now,

$$\frac{1}{F(r)} = \frac{1}{H(r_0)(r-r_0)} + \frac{1}{F(r)} - \frac{1}{H(r_0)(r-r_0)} = \frac{1}{H(r_0)(r-r_0)} + \frac{H(r_0)-H(r)}{H(r)H(r_0)(r-r_0)}\,.$$

An analysis of $H(r) - H(r_0)$ as in (1.2.47) allows us to integrate the equation $f' = 1/F$ in the form

$$f(r) = \frac{1}{F'(r_0)}\ln|r-r_0| + \hat{f}(r)\,,$$

for some function $\hat{f}$ which is smooth near r_0. Inserting all this into (1.2.46) with

$$c = \frac{F'(r_0)}{2}$$

gives

$$g = \mp\frac{4H(r)}{(F'(r_0))^2}\exp(-\hat{f}(r)F'(r_0))d\hat{u}\,d\hat{v} + h\,, \quad (1.2.48)$$

with a negative sign if we started in the region $r > r_0$, and positive otherwise.

The function r is again implicitly defined by the equation

$$\hat{u}\hat{v} = \mp(r-r_0)\exp(\hat{f}(r)F'(r_0))\,.$$

The right-hand side has a derivative which equals $\mp\exp(\hat{f}(r_0)/F'(r_0)) \neq 0$ at r_0, and therefore this equation defines a smooth function $r = r(\hat{u}\hat{v})$ for r near r_0 by the implicit function theorem.

The above discussion applies to F which are of C^k differentiability class, with some losses of differentiability. Indeed, (1.2.48) provides an extension of C^{k-2} differentiability class, which leads to the restriction $k \geq 2$. However, the implicit function argument just given requires h to be differentiable, so we need in fact $k \geq 3$ for a coherent analysis. Note that for real analytic F's the extension so constructed is real analytic; this follows from the analytic version of the implicit function theorem.

Supposing we start with a region where $r > r_0$, with F positive there. Then we are in a situation reminiscent of that we encountered with the Schwarzschild metric, where a single region of the type I in Figure 1.2.1 leads to the attachment of *three* new regions to the initial manifold, through "a lower left horizon, and an upper left horizon, meeting at a corner". On the other hand, if we start with $r < r_0$ and F is negative there, we are in the situation of Figure 1.2.1 where a region of type II is extended through "an upper left horizon, and an upper right horizon, meeting at a corner". The reader should have no difficulties examining all remaining possibilities. We return to this in Chapter 4. □

The function f of (1.2.44) for a (4+1)–dimensional Schwarzschild-Tangherlini solution can be calculated to be

$$f = r + \sqrt{2m}\ln\Big(\frac{r-\sqrt{2m}}{r+\sqrt{2m}}\Big)\,.$$

A direct calculation leads to

$$g = -\frac{32m(r+\sqrt{2m})^2}{r^2}\exp(-r/2m)\,d\hat{u}\,d\hat{v} + d\Omega^2\,. \tag{1.2.49}$$

One can similarly obtain (non-very-enlightening) explicit expressions in dimension $(5+1)$.

The isotropic coordinates in higher dimensions lead to the following form of the Schwarzschild-Tangherlini metric [231]:

$$g_m = \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}}\left(\sum_{1=1}^{n}(dx^i)^2\right) - \left(\frac{1-m/2|x|^{n-2}}{1+m/2|x|^{n-2}}\right)^2 dt^2\,. \tag{1.2.50}$$

The radial coordinate $|x|$ in (1.2.50) is related to the radial coordinate r of (1.2.42) by the formula

$$r = \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{2}{n-2}}|x|\,.$$

It may be considered unsatisfactory that the function r appearing in the globally regular form of the metric (1.2.25) is not given by an explicit elementary function of the coordinates. Here is a an explicit form of the extended Schwarzschild metric due to Israel [159][7]

$$g = -8m\left[dxdy + \frac{y^2}{xy+2m}dx^2\right] - (xy+2m)^2 d\Omega^2\,. \tag{1.2.51}$$

The coordinates (x,y) are related to the standard Schwarzschild coordinates (t,r) as follows:

$$\begin{aligned}
r &= xy+2m\,, && (1.2.52)\\
t &= xy+2m(1+\ln|y/x|)\,, && (1.2.53)\\
|x| &= \sqrt{|r-2m|}\exp\left(\frac{r-t}{4m}\right)\,, && (1.2.54)\\
|y| &= \sqrt{|r-2m|}\exp\left(\frac{t-r}{4m}\right)\,. && (1.2.55)
\end{aligned}$$

In higher dimensions one also has an explicit, though again not very enlightening, manifestly globally regular form of the metric [182], in spacetime dimension $n+1$:

$$\begin{aligned}
ds^2 &= -2\frac{w^2(-(r)^{-n+2}2^{n+1}m^{n+1}+4m^2((n+1)(2m-r)+3r-4m)}{m(2m-r)^2}dU^2\\
&\quad +8mdUdw + r^2 d\Omega^2_{n-1}\,, && (1.2.56)
\end{aligned}$$

where $r\geq 0$ is the function

$$r(U,w) \equiv 2\,m + (n-2)Uw, \tag{1.2.57}$$

while $d\Omega^2_{n-1}$ is the metric of a unit round $n-1$ sphere.

[7]The Israel coordinates have been found independently in [228], see also [171].

1.2.5 Some geodesics

The geodesics in the Schwarzschild metric have been studied extensively in the literature (cf., e.g., [51]), so we will only make a few general comments about those.

First, we already encountered a family of outgoing and incoming radial null geodesics $t \mp (r + 2m\ln(r-2m)) = \text{const}$.

Next, we have seen that the horizon $\{r = 2m\}$ is threaded by a family of null geodesics, its generators.

We continue by noting that each Killing vector X produces a constant of motion $g(X, \dot\gamma)$ along an affinely parameterised geodesic. So we have a conserved energy-per-unit-mass

$$\mathscr{E} := g(\partial_t, \dot\gamma) = -(1 - \frac{2m}{r})\dot t\,,$$

and a conserved angular-momentum-per-unit-mass J

$$J := g(\partial_\varphi, \dot\gamma) = r^2\dot\varphi\,.$$

Yet another constant of motion arises from the length of $\dot\gamma$,

$$g(\dot\gamma, \dot\gamma) = -(1 - \frac{2m}{r})\dot t^2 + \frac{\dot r^2}{1 - \frac{2m}{r}} + r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = \varepsilon \in \{-1, 0, 1\}\,. \quad (1.2.58)$$

INCIDENTALLY: To simplify things somewhat, let us show that all motions are *planar*. One way of doing this is to write the equations explicitly. The Lagrangian for geodesics reads:

$$\mathscr{L} = \frac{1}{2}\left(-V^2\left(\frac{dt}{ds}\right)^2 + V^{-2}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right)\,.$$

Those Euler-Lagrange equations which are not already covered by the conservation laws read:

$$\begin{aligned}
\frac{d}{ds}\left(V^{-2}\frac{dr}{ds}\right) &= -\partial_r V\left(V\left(\frac{dt}{ds}\right)^2 + V^{-3}\left(\frac{dr}{ds}\right)^2\right) \\
&\quad + r\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right]\,, \qquad (1.2.59) \\
\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) &= r^2\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2\,. \qquad (1.2.60)
\end{aligned}$$

Consider any geodesic, and think of the coordinates (r, θ, φ) as spherical coordinates on $\mathbb{R}^3$. Then the initial position vector (which is, for obvious reasons, assumed *not* to be the origin) and the initial velocity vector, which is assumed *not* to be radial (otherwise the geodesic will be radial, and the claim follows) define a unique plane in $\mathbb{R}^3$. We can then choose the spherical coordinates so that this plane is the plane $\theta = \pi/2$. We then have $\theta(0) = \pi/2$ and $\dot\theta(0) = 0$, and then $\theta(s) \equiv \pi/2$ is a solution of (1.2.60) satisfying the initial values. By uniqueness this is *the* solution. □

So, without loss of generality we can assume $\sin\theta = 1$ throughout the motion, from (1.2.58) we then obtain the following ODE for $r(s)$:

$$\dot{r}^2 = \mathscr{E}^2 + (1 - \frac{2m}{r})(\varepsilon - \frac{J^2}{r^2})\,. \tag{1.2.61}$$

The radial part of the geodesic equation can be obtained by calculating directly the Christoffel symbols of the metric. A more efficient way is to use the variational principle for geodesics, with the Lagrangian $\mathscr{L} = g(\dot\gamma, \dot\gamma)$ — this can be read off from the middle term in (1.2.58). But the reader should easily convince herself that, at this stage, the desired equation can be obtained by differentiating (1.2.61) with respect to s, obtaining

$$2\frac{d^2r}{ds^2} = \frac{d}{dr}\Big(\mathscr{E}^2 + (1 - \frac{2m}{r})(\varepsilon - \frac{J^2}{r^2})\Big)\,. \tag{1.2.62}$$

We wish to point out the existence of a striking class of *null* geodesics for which $r(s) = \text{const}$. It follows from (1.2.62), and from uniqueness of solutions of the Cauchy problem for ODE's, that such a curve will be a null geodesic provided that the right-hand sides of (1.2.61) and of (1.2.62) (with $\varepsilon = 0$) vanish:

$$\mathscr{E}^2 - (1 - \frac{2m}{r})\frac{J^2}{r^2} = 0 = \frac{2J^2}{r^3}(-r + 3m)\,. \tag{1.2.63}$$

Simple algebra shows now that the curves

$$s \mapsto \gamma_\pm(s) = (t = s,\ r = 3m,\ \theta = \pi/2,\ \varphi = \pm 3^{3/2} m^{-1} s)\,, \tag{1.2.64}$$

are null geodesics spiraling on the timelike cylinder $\{r = 3m\}$.

EXERCICE 1.2.14 Let γ be a timelike geodesic for the Schwarzschild metric parameterized by proper time and lying in the equatorial plane $\theta = \pi/2$. Show that

$$\frac{E^2 - \dot r^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1\,.$$

Deduce that if $E = 1$ and $J = 4m$ then

$$\frac{\sqrt{r} - 2\sqrt{m}}{\sqrt{r} + 2\sqrt{m}} = A e^{\epsilon\varphi/\sqrt{2}}\,,$$

where $\epsilon = \pm 1$ and A is a constant. Describe the orbit that starts at $\varphi = 0$ in each of the cases (i) $A = 0$, (ii) $A = 1$, $\epsilon = -1$, (iii) $r(0) = 3m$, $\epsilon = -1$. □

EXERCICE 1.2.15 Let $u = m/r$. Show that there exist constants E, J and λ such that along non-radial geodesics we have

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - \big(u^2 + \frac{\lambda m^2}{J^2}\big)(1 - 2u)\,. \tag{1.2.65}$$

Show that for every $r > 3m$ there exist timelike geodesics for which $\dot r = 0$.

Consider a geodesic which is a small perturbation of a fixed-radius geodesic. Writing $u = u_0 + \delta u$, where $du_0/d\varphi = 0$, and where δu is assumed to be small, derive a linear second order differential equation approximatively satisfied by δu. Solving this equation, conclude that for $3m < r < 6m$ the constant-radius geodesics are unstable at a linearized level, while they are linearization-stable for $r > 6m$. □

EXERCICE 1.2.16 Consider (1.2.65) with $\lambda = 0$ and $J = 3\sqrt{3}mE$. Check that we have then

$$\left(\frac{du}{d\varphi}\right)^2 = 2\left(u+\frac{1}{6}\right)\left(u-\frac{1}{3}\right)^2 , \tag{1.2.66}$$

and that for any $\varphi_0 \in \mathbb{R}$ the function

$$u(\varphi) = -\frac{1}{6} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(\varphi-\varphi_0)\right] \tag{1.2.67}$$

solves (1.2.66). Study the asymptotic behaviour of these solutions. What can you infer from (1.2.67) about stability of the spiraling geodesics (1.2.64)? □

Insightful animations of ray tracing in the Schwarzschild spacetime can be found at `http://jilawww.colorado.edu/~ajsh/insidebh/schw.html`.

1.2.6 The Flamm paraboloid

We write again the Schwarzschild metric in dimension $n+1$,

$$g_m = -\left(1-\frac{2m}{r^{n-2}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r^{n-2}}} + r^2 d\Omega^2 , \tag{1.2.68}$$

where, as usual, $d\Omega^2$ is the round unit metric on S^{n-1}. Because of spherical symmetry, the geometry of the $t = \text{const}$ slices can be realised by an embedding into $(n+1)$–dimensional Euclidean space. If we set

$$\mathring{g} = dz^2 + (dx^1)^2 + \ldots + (dx^n)^2 = dz^2 + dr^2 + r^2 d\Omega^2 ,$$

the metric h induced by $\mathring{g}$ on the the surface $z = z(r)$ reads

$$h = \left(\left(\frac{dz}{dr}\right)^2 + 1\right)dr^2 + r^2 d\Omega^2 . \tag{1.2.69}$$

This will coincide with the space part of (1.2.68) if we require that

$$\frac{dz}{dr} = \pm\sqrt{\frac{2m}{r^{n-2}-2m}} .$$

The equation can be explicitly integrated in dimensions $n = 3$ and 4 in terms of elementary functions, leading to

$$z = z_0 \pm \sqrt{2m} \times \begin{cases} 2\sqrt{r-2m}, & r > 2m,\ n = 3, \\ \ln(r+\sqrt{r^2-2m}), & r > \sqrt{2m},\ n = 4. \end{cases}$$

The positive sign corresponds to the usual black hole exterior, while the negative sign corresponds to the second asymptotically flat region, on the “other side” of the *Einstein-Rosen bridge*. Solving for $r(z)$, a convenient choice of z_0 leads to

$$r = \begin{cases} 2m + z^2/8m, & n = 3, \\ \sqrt{2m}\cosh(z/\sqrt{2m}), & n = 4. \end{cases}$$

In dimension $n = 3$ one obtains a paraboloid, as first noted by Flamm. The embeddings are visualized in Figures 1.2.2 and 1.2.3.

The qualitative behavior in dimensions $n \geq 5$ is somewhat different, as then $z(r)$ asymptotes to a finite value as r tends to infinity. The embeddings in $n = 5$ are visualized in Figure 1.2.4; in that dimension $z(r)$ can be expressed in terms of elliptic functions, but the final formula is not very illuminating.

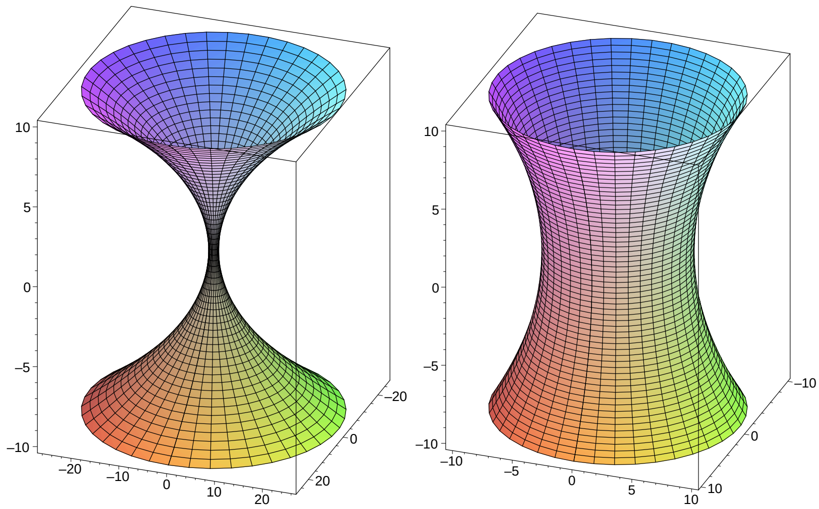

Figure 1.2.2: Isometric embedding of the space-geometry of an $n = 3$ dimensional Schwarzschild black hole into four-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = 2m$, with $2m = 1$ (left) and $2m = 6$ (right).

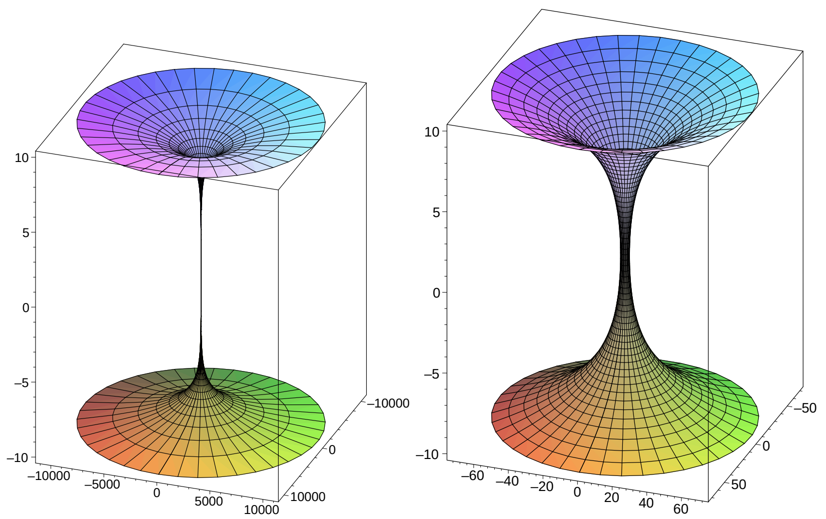

Figure 1.2.3: Isometric embedding of the space-geometry of an $n = 4$ dimensional Schwarzschild black hole into five-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/2}$, with $2m = 1$ (left) and $2m = 6$ (right). The extents of the vertical axes are the same as those in Figure 1.2.2.

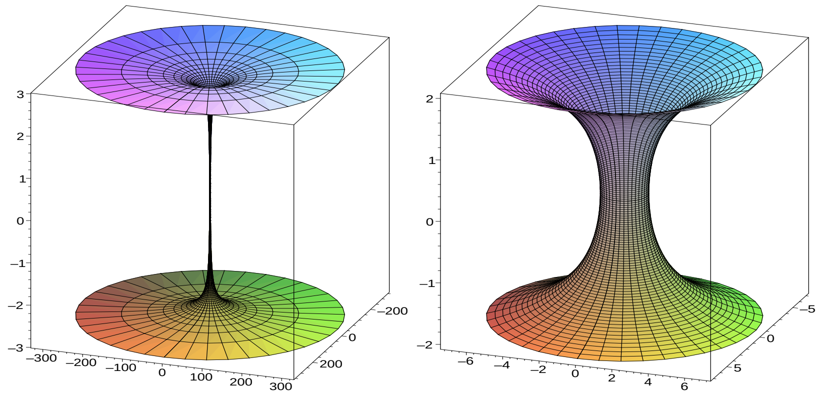

Figure 1.2.4: Isometric embedding of the *space-geometry* of a (5 + 1)–dimensional Schwarzschild black hole into six-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/3}$, with $2m = 2$. The variable along the vertical axis asymptotes to $\approx \pm 3.06$ as r tends to infinity. The right picture is a zoom to the centre of the throat.

1.2.7 Fronsdal's embedding

An embedding of the full Schwarzschild geometry into six dimensional Minkowski spacetime has been constructed by Fronsdal [124] (compare [107–109]). For this, let us write the flat metric η on $\mathbb{R}^6$ as

$$\eta = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 \, .$$

For $r > 2m$ the required embedding is obtained by setting

$$\begin{gathered} z^0 = 4m\sqrt{1-2m/r}\,\sinh(t/4m)\,, \qquad z^1 = 4m\sqrt{1-2m/r}\,\cosh(t/4m)\,, \\ z^2 = \int \sqrt{2m(r^2+2mr+4m^2)/r^3}dr\,, \qquad (1.2.70) \\ z^3 = r\sin\theta\sin\phi\,, \quad z^4 = r\sin\theta\cos\phi\,, \quad z^5 = r\cos\phi\,. \end{gathered}$$

(The function z^2, plotted in Figure 1.2.5, can be found explicitly in terms of elliptic integrals, but the final formula is not very enlightening.) The embedding

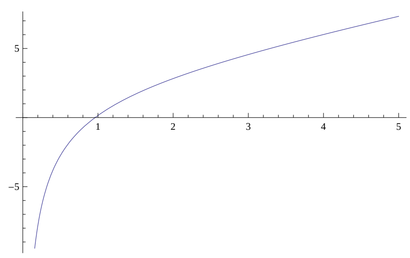

Figure 1.2.5: The function z^2/m of (1.2.70) in terms of r/m.

is visualised in Figure 1.2.6. Note that z^2 is defined and analytic for all $r > 0$,

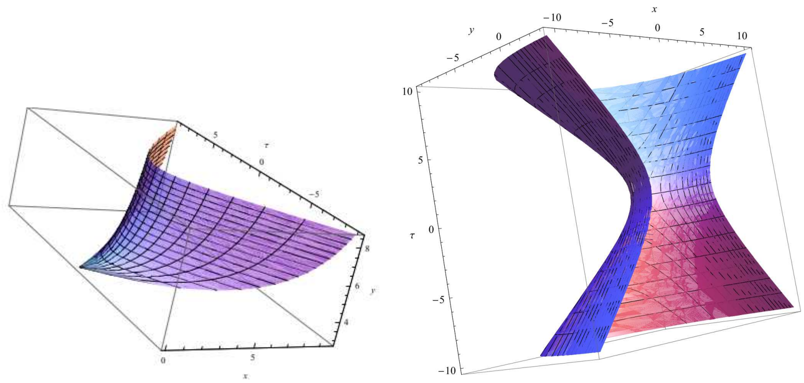

Figure 1.2.6: Fronsdal's embedding $(t,r) \mapsto (x = z^1, \tau = z^0, y = z^2)$, with target metric $-d\tau^2 + dx^2 + dy^2$, of the region $r > 2m$ (left figure) and of the whole Kruskal-Szekeres manifold (right figure) with $m = 1$.

which allows one to extend the map (1.2.70) analytically to the whole Kruskal-Szekeres manifold. This led Fronsdal to his own discovery of the Kruskal-Szekeres extension of the Schwarzschild metric, but somehow his name is rarely mentioned in this context.

Exercice 1.2.17 Show that the formulae

$$z^0 = 4m\sqrt{1-2m/r^{n-2}}\sinh(t/4m)\,, \qquad z^1 = 4m\sqrt{1-2m/r^{n-2}}\cosh(t/4m)\,,$$

$$z^2 = \sqrt{2}\sqrt{\frac{mr^{2-n}\left(r^n - 8m^3(n-2)^2\right)}{r^n - 2mr^2}} \tag{1.2.71}$$

can be used to construct an embedding of $(n+1)$-dimensional Schwarzschild metric with $n \geq 3$ into $\mathbb{R}^{1,n+2}$. □

Exercice 1.2.18 Prove that no embedding of $(n+1)$-dimensional Schwarzschild metric with $n \geq 3$ into $\mathbb{R}^{1,n+1}$ exists. □

Exercice 1.2.19 Find an embedding of the Schwarzschild metric into $\mathbb{R}^6$ with metric $\eta = -(dz^0)^2 - (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2$. You may wish to assume $z^0 = f(r)\cos(t/4m)$, $z^1 = f(r)\sin(t/4m)$, $z^2 = h(r)$, with the remaining functions as in (1.2.70). □

1.2.8 Conformal Carter-Penrose diagrams

Consider a metric with the following product structure:

$$g = \underbrace{g_{rr}(t,r)dr^2 + 2g_{rt}(t,r)dtdr + g_{tt}(t,r)dt^2}_{=:{}^2g} + \underbrace{h_{AB}(t,r,x^A)dx^A dx^B}_{=:h}\,, \tag{1.2.72}$$

where h is a Riemannian metric in dimension $n-1$. Then any causal vector for g is also a causal vector for ${}^2g$, and drawing light-cones for ${}^2g$ gives a good idea of the causal structure of $(\mathscr{M}, g)$. We have already done that in Figure 1.2.1 to depict the black hole character of the Kruskal-Szekeres spacetime.

Now, it is not too difficult to prove that any two-dimensional Lorentzian metric can be locally written in the form

$$^2g = 2g_{uv}(u,v)dudv = 2g_{uv}(-dt^2 + dr^2) \tag{1.2.73}$$

in which the light-cones have slopes one, just as in Minkowski spacetime. When using such coordinates, it is sufficient to draw their domain of definition to visualise the global causal structure of the spacetime.

EXERCICE 1.2.20 Prove (1.2.73). *[Hint: introduce coordinates associated with right-going and left-going null geodesics.]* □

The above are the first two-ingredients behind the idea of conformal Carter-Penrose diagrams. The last thing to do is to bring any infinite domain of definition of the (u,v) coordinates to a finite one. We will discuss this how to do quite generally in Chapter 4, but it is of interest to do it explicitly for the Kruskal-Szekeres spacetime. For this, let $\bar u$ and $\bar v$ be defined by the equations

$$\tan \bar u = \hat u\,, \quad \tan \bar v = \hat v\,,$$

where $\hat v$ and $\hat u$ have been defined in (1.2.23)-(1.2.24). Using

$$d\hat u = \frac{1}{\cos^2 \bar u} d\bar u\,, \quad d\hat v = \frac{1}{\cos^2 \bar v} d\bar v\,,$$

the Schwarzschild metric (1.2.25) takes the form

$$\begin{aligned} g &= -\frac{32m\exp(-\frac{r}{2m})}{r} d\hat u\, d\hat v + r^2 d\Omega^2 \\ &= -\frac{32m\exp(-\frac{r}{2m})}{r\cos^2 \bar u \cos^2 \bar v} d\bar u\, d\bar v + r^2 d\Omega^2\,. \end{aligned} \tag{1.2.74}$$

Introducing new time- and space-coordinates $\bar t = (\bar u + \bar v)/2$, $\bar x = (\bar v - \bar u)/2$, so that

$$\bar u = \bar t - \bar x\,, \quad \bar v = \bar t + \bar x\,,$$

one obtains a more familiar-looking form

$$g = \frac{32m\exp(-\frac{r}{2m})}{r\cos^2 \bar u \cos^2 \bar v}(-d\bar t^2 + d\bar x^2) + r^2 d\Omega^2\,.$$

This is regular except at $\cos \bar u = 0$, or $\cos \bar v = 0$, or $r = 0$. The first set corresponds to the straight lines $\bar u = \bar t - \bar x \in \{\pm\pi/2\}$, while the second is the union of the lines $\bar v = \bar t + \bar x \in \{\pm\pi/2\}$.

The analysis of $\{r = 0\}$ requires some work: recall that $r \to 0$ corresponds to $\hat u \hat v \to 1$, which is equivalent to

$$\tan(\bar u)\tan(\bar v) \to 1\,.$$

Using the formula

$$\tan(\bar u + \bar v) = \frac{\tan \bar u + \tan \bar v}{1 - \tan \bar u \tan \bar v}$$

we obtain $\tan(\bar u + \bar v) \to_{r\to 0} \pm\infty$ unless perhaps the numerator tends to zero. Except for the last borderline cases, this is equivalent to

$$\bar u + \bar v = 2\bar t \to \pm\pi/2\,.$$

So the Kruskal-Szekeres metric is conformal to a smooth Lorentzian metric on $C \times S^2$, where C is the set of Figure 1.2.7.

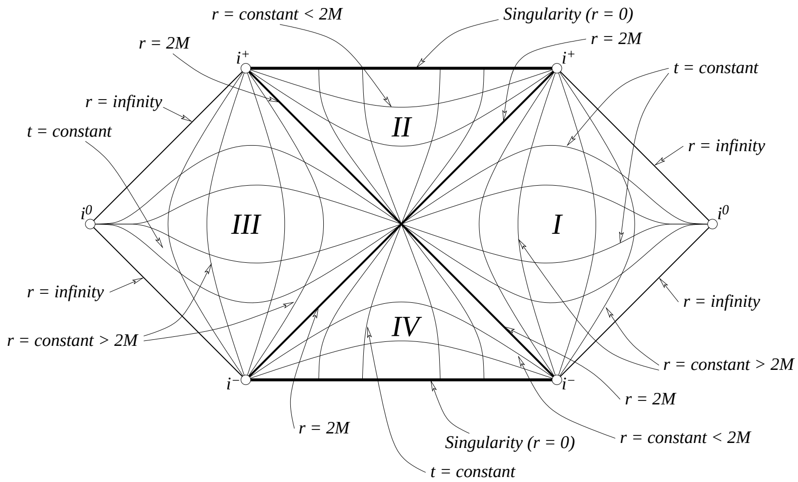

Figure 1.2.7: The Carter-Penrose diagram[5] for the Kruskal-Szekeres spacetime with mass M. There are actually two asymptotically flat regions, with corresponding event horizons defined with respect to the second region. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one. Regions are numbered as in Figure 1.2.1.

1.2.9 Weyl coordinates

A set of coordinates well suited to study static axisymmetric metrics has been introduced by Weyl. In those coordinates the Schwarzschild metric takes the form (*cf.*, *e.g.,* [259, Equation (20.12)])

$$g = -e^{2U_{\rm Schw}}dt^2 + e^{-2U_{\rm Schw}}\rho^2 d\varphi^2 + e^{2\lambda_{\rm Schw}}(d\rho^2+dz^2)\,, \tag{1.2.75}$$

where

$$\begin{aligned} U_{\rm Schw} &= \ln\rho - \ln\left(m\sin\tilde\theta + \sqrt{\rho^2+m^2\sin^2\tilde\theta}\right) && (1.2.76)\\ &= \frac{1}{2}\ln\left[\frac{\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}-2m}{\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}+2m}\right]\,, && (1.2.77)\\ \lambda_{\rm Schw} &= -\frac{1}{2}\ln\left[\frac{(r_{\rm Schw}-m)^2-m^2\cos^2\tilde\theta}{r^2_{\rm Schw}}\right] && (1.2.78)\\ &= -\frac{1}{2}\ln\left[\frac{4\sqrt{(z-m)^2+\rho^2}\sqrt{(z+m)^2+\rho^2}}{\left[2m+\sqrt{(z-m)^2+\rho^2}+\sqrt{(z+m)^2+\rho^2}\right]^2}\right] && (1.2.79)\end{aligned}$$

In (1.2.76) the angle $\tilde\theta$ is a Schwarzschild angular variable, with the relations

$$\begin{aligned} 2m\cos\tilde\theta &= \sqrt{(z+m)^2+\rho^2}-\sqrt{(z-m)^2+\rho^2}\,,\\ 2(r_{\rm Schw}-m) &= \sqrt{(z+m)^2+\rho^2}+\sqrt{(z-m)^2+\rho^2}\,,\\ \rho^2 = r_{\rm Schw}(r_{\rm Schw}-2m)\sin^2\tilde\theta\,,&\quad z=(r_{\rm Schw}-m)\cos\tilde\theta\,,\end{aligned}$$

where r_{Schw} is the usual Schwarzschild radial variable such that $e^{2U_{\mathrm{Schw}}} = 1 - 2m/r_{\mathrm{Schw}}$. As is well known, and in any case easily seen, U_{Schw} is smooth on $\mathbb{R}^3$ except on the set $\{\rho = 0, -m \le z \le m\}$. From (1.2.76) we find, at fixed z in the interval $-m < z < m$ and for small ρ,

$$U_{\mathrm{Schw}}(\rho, z) = \ln\rho - \ln(2\sqrt{(m+z)(m-z)}) + O(\rho^2) \tag{1.2.80}$$

(with the error term *not* uniform in z).

The value $\mathring{\lambda}$ of λ on the rod equals

$$\mathring{\lambda}(z) = -\frac{1}{2}\ln\left[\frac{(m-z)(z+m)}{(2m)^2}\right].$$

1.3 Some general notions

1.3.1 Isometries

Before continuing some general notions are in order. A *Killing field*, by definition, is a vector field the local flow of which preserves the metric. Equivalently, X satisfies the *Killing equation*,

$$0 = \mathscr{L}_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu\,. \tag{1.3.1}$$

The set of solutions of this equation forms a Lie algebra, where the bracket operation is the bracket of vector fields (see Section A.21, p. 305).

One of the features of the Schwarzschild metric (1.2.1) is its *stationarity*, with Killing vector field $X = \partial_t$: A spacetime is called *stationary* if there exists a Killing vector field X which approaches ∂_t in the asymptotically flat region (where r goes to ∞, see Section 1.3.6 below for precise definitions) *and* generates a one parameter groups of isometries. A spacetime is called *static* if it is stationary and if the stationary Killing vector X is hypersurface-orthogonal, i.e.

$$X^\flat \wedge dX^\flat = 0\,, \tag{1.3.2}$$

where

$$X^\flat = X_\mu dx^\mu = g_{\mu\nu}X^\nu dx^\mu\,.$$

Incidentally: Exercice 1.3.2 Show that the Schwarzschild metric, as well as the Reissner-Nordström metrics of Section 1.5, are static but the Kerr metrics with $a \neq 0$, presented in Section 1.6 below, are not.

□

Any metric with a Killing vector field X can be locally written, away from the zeros of X, in the form

$$g = -V(dt + \underbrace{\theta_i dx^i}_{=:\theta})^2 + h_{ij}dx^i dx^j\,, \tag{1.3.3}$$

with $X=\partial_t$ and hence $\partial_t g_{\mu\nu}=0$. Then

$$X^\flat = -V(dt+\theta)\,,\quad X^\flat = -dV\wedge(dt+\theta)-V d\theta\,,\quad X^\flat\wedge dX^\flat = V^2(dt+\theta)\wedge d\theta\,. \tag{1.3.4}$$

It follows that g is static if and only if $d\theta=0$. Therefore, for static metrics, on any simply connected subset of M there exists a function $f=f(x^i)$ such that $\theta=df$. Introducing a new time coordinate $\tau := t+f$, we conclude that any static metric g can locally be written as

$$g=-Vd\tau^2+h_{ij}dx^idx^j\,. \tag{1.3.5}$$

We also see that staticity leads to, and is equivalent to, the existence of a supplementary local discrete isometry of g obtained by mapping τ to its negative, $\tau\mapsto-\tau$.

On a simply connected spacetime M the representation (1.3.5) is global, with a function V without zeros, provided that there exists in M a hypersurface $\mathscr{S}$ which is transverse to a globally timelike Killing vector X, with every orbit of X meeting $\mathscr{S}$ precisely once.

A spacetime is called *axisymmetric* if there exists a Killing vector field Y, which generates a one parameter group of isometries, and which behaves like a *rotation*: this property is captured by requiring that all orbits 2π periodic, and that the set $\{Y=0\}$, called the *axis of rotation*, is non-empty. Killing vector fields which are a non-trivial linear combination of a time translation and of a rotation in the asymptotically flat region are called *stationary-rotating*, or *helical*. Note that those definitions require completeness of orbits of all Killing vector fields (this means that the equation $\dot x = X$ has a global solution for all initial values), see [60] and [130] for some results concerning this question.

In the extended Schwarzschild spacetime the set $\{r=2m\}$ is a null hypersurface $\mathscr{E}$, the Schwarzschild event horizon. The stationary Killing vector $X=\partial_t$ extends to a Killing vector $\hat X$ in the extended spacetime which becomes tangent to and null on $\mathscr{E}$, except at the "bifurcation sphere" right in the middle of Figure 1.2.7, where $\hat X$ vanishes.

1.3.2 Killing horizons

A *null* hypersurface which coincides with a connected component of the set

$$\mathscr{N}_X := \{g(X,X)=0\,,\ X\neq 0\}\,, \tag{1.3.6}$$

where X is a Killing vector, with X *tangent* to $\mathscr{N}$, is called a *Killing horizon* associated to X. Here it is implicitly assumed that the hypersurface is *embedded*.

We will sometimes write $\mathscr{N}(X)$ instead of $\mathscr{N}_X$.

Example 1.3.3 The simplest example is provided by the "boost Killing vector field"

$$X=z\partial_t+t\partial_z \tag{1.3.7}$$

in Minkowski spacetime: The Killing horizon $\mathscr{N}_X$ of X has four connected components

$$\mathscr{N}(X)_{\epsilon\delta} := \{t=\epsilon z\,,\delta t>0\}\,,\quad \epsilon,\delta\in\{\pm1\}\,; \tag{1.3.8}$$

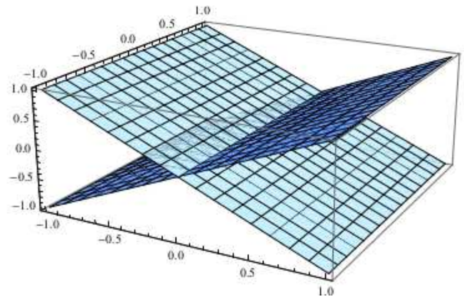

Figure 1.3.1: The four branches of a bifurcate horizon and the bifurcation surface for the boost Killing vector $x\partial_t + t\partial_x$ in three-dimensional Minkowski spacetime.

indeed, we have

$$\{g(X,X)=0\}=\{t=\pm z\}\,, \tag{1.3.9}$$

but from this we need to remove the set of points $\{z = t = 0\}$, where X vanishes. The closure $\overline{\mathscr{N}_X}$ of $\mathscr{N}_X$ is the set $\{|t| = |z|\}$, as in (1.3.9), which is *not* a manifold, because of the crossing of the null hyperplanes $\{t = \pm z\}$ at $t = z = 0$; see Figure 1.3.1. Horizons of this type are referred to as *bifurcate Killing horizons*. More precisely, a set will be called a *bifurcate Killing horizon* if it is the union of a smooth submanifold S of co-dimension two, called the *bifurcation surface*, and of four Killing horizons obtained by shooting null geodesics in the four distinct null directions orthogonal to S. So, the Killing vector $z\partial_t + t\partial_z$ in Minkowski spacetime has a bifurcate Killing horizon, with the bifurcation surface $\{t = z = 0\}$.

EXAMPLE 1.3.4 Figure 1.2.1 on p. 18 makes it clear that the set $\{r = 2m\}$ in the Kruskal-Szekeres spacetime is the union of four Killing horizons and of the bifurcation surface, with respect to the Killing vector field which equals ∂_t in the asymptotically flat region.

It turns out that the above examples are typical. Indeed, consider a spacelike submanifold S of co-dimension two in a spacetime $(\mathscr{M}, g)$, and suppose that there exists a (non-trivial) Killing vector field X which vanishes on S. Then the one-parameter group of isometries $\phi_t[X]$ generated by X leaves S invariant and, along S, the tangent maps $\phi_t[X]_*$ induce isometries of $T\mathscr{M}$ to itself. At every $p \in S$ there exist precisely two null directions $\mathrm{Vect}\{n_\pm\} \subset T_p\mathscr{M}$, where $n_\pm$ are two distinct null future directed vectors normal to S. Since every geodesic is uniquely determined by its initial point and its initial direction, we conclude that the null geodesics through p are mapped to themselves by the flow of X. Thus X is tangent to those geodesics. There exist two null hypersurfaces $\mathscr{N}_\pm$ threaded by those null geodesics, intersecting at S. We define $\mathscr{N}_{\pm+}$ to be the connected components of $\mathscr{N}_\pm \setminus \{X = 0\}$ lying to the *future* of S and accumulating at S. Similarly we define $\mathscr{N}_{\pm-}$ to be the connected components of $\mathscr{N}_\pm \setminus \{X = 0\}$ lying to the *past* of S and accumulating at S. Then the $\mathscr{N}_{\pm\pm}$

are Killing horizons which, together with S, form a bifurcate Killing horizon with bifurcation surface S.

EXAMPLE 1.3.5 One more noteworthy example, in Minkowski spacetime, is provided by the Killing vector

$$X = y\partial_t + t\partial_y + x\partial_y - y\partial_x = y\partial_t + (t+x)\partial_y - y\partial_x \,. \tag{1.3.10}$$

Thus, X is the sum of a boost $y\partial_t + t\partial_y$ and a rotation $x\partial_y - y\partial_x$. Note that X vanishes if and only if

$$y = t + x = 0\,,$$

which is a two-dimensional isotropic (null) submanifold of Minkowski spacetime $\mathbb{R}^{1,3}$. Further,

$$g(X,X) = (t+x)^2 = 0$$

which is an isotropic hyperplane in $\mathbb{R}^{1,3}$.

REMARK 1.3.6 When attempting to prove uniqueness of black holes, one is naturally led to the following notion: Let X be a Killing vector, then every connected, not necessarily embedded, null hypersurface $\mathscr{N}_0 \subset \mathscr{N}_X$, with $\mathscr{N}_X$ as in (1.3.6), with the property that X is tangent to $\mathscr{N}_0$, is called a *Killing prehorizon*.

One of the fundamental differences between prehorizons and horizons is that the latter are necessary embedded, while the former are allowed not to be. Thus, a Killing horizon is also a Killing prehorizon, but the reverse implication is not true. As an example, consider $\mathbb{R} \times \mathbb{T}^2$ with the flat product metric, and let Y be any covariantly constant unit vector on $\mathbb{T}^2$ the orbits of which are dense on $\mathbb{T}^2$. Let $\Gamma \subset \mathbb{T}^2$ be such an orbit, then $\mathbb{R} \times \Gamma$ provides an example of non-embedded prehorizon associated with the null Killing vector $X := \partial_t + Y$.

Prehorizons are a major headache to handle in analytic arguments, and one of the key steps of the uniqueness theory of stationary black holes is to prove that they do not exist within the domain of outer commmunications of well behaved black-hole spacetimes [71, 75, 198]. □

1.3.3 Surface gravity

The *surface gravity* κ of a Killing horizon $\mathscr{N}(X)$ is defined by the formula

$$(X^\alpha X_\alpha)_{,\mu}\Big|_{\mathscr{N}(X)} = -2\kappa X_\mu \,. \tag{1.3.11}$$

A word of justification is in order here: since $g(X,X)=0$ on $\mathscr{N}(X)$ the differential of $g(X,X)$ is conormal to $\mathscr{N}(X)$. (A form α is said to be *conormal* to S if for every vector $Y \in TS$ we have $\alpha(Y) = 0$.) Recalling (cf. Appendix A.23, p. 316) that on a null hypersurface the conormal is proportional to $g(\ell,\cdot)$, where ℓ is any null vector tangent to $\mathscr{N}$ (those are defined uniquely up to a proportionality factor), we obtain that $d(g(X,X))$ is proportional to $X^\flat = X_\mu dx^\mu$; whence (1.3.11). We will show shortly that κ is a constant under fairly general conditions.

The surface gravity of black holes plays an important role in *black hole thermodynamics*, *cf. e.g.,* [34] and references therein.

REMARK 1.3.7 The surface gravity measures the acceleration a of the integral curves of the Killing vector X, in the following sense: Let $\tau \mapsto x^\mu(\tau)$ be an integral curve of X, thus $x^\mu(\tau)$ solves the equation

$$\dot x^\mu(\tau) \equiv \frac{dx^\mu}{d\tau}(\tau) = X^\mu(x^\alpha(\tau))\,.$$

The acceleration $a = a^\mu\partial_\mu$ of the curve $x^\mu(\tau)$ is defined as $D\dot x^\mu/d\tau$. On the Killing horizon $\mathscr{N}_X$ we have

$$\begin{aligned} a^\mu &= \frac{D\dot x^\mu}{d\tau} = \dot x^\sigma\nabla_\sigma X^\mu = X^\sigma\nabla_\sigma X^\mu = -X^\sigma\nabla^\mu X_\sigma = -\frac{1}{2}\nabla^\mu(X^\sigma X_\sigma) \\ &= \kappa\dot x^\mu = \kappa X^\mu\,. \end{aligned}$$

Thus

$$a = \kappa X \tag{1.3.12}$$

on the Killing horizon. □

As an example of calculation of surface gravity, consider the Killing vector X of (1.3.7). We have

$$d(g(X,X)) = d(-z^2+t^2) = 2(-zdz+tdt)\,.$$

On $\mathscr{N}(X)_{\epsilon\delta}$ we have $t=\epsilon z$, and

$$X^\flat = -zdt+tdz = z(-dt+\epsilon dz) = -\frac{1}{2}\epsilon d(g(X,X))|_{\mathscr{N}(X)_{\epsilon\delta}}\,,$$

and so

$$\kappa = \epsilon \in \{\pm 1\}\,. \tag{1.3.13}$$

As another example, for the Killing vector X of (1.3.10) we have

$$d(g(X,X)) = 2(t+x)(dt+dx)\,,$$

which vanishes on each of the Killing horizons $\{t=-x\,, y\neq 0\}$. We conclude that $\kappa = 0$ on both horizons.

A Killing horizon $\mathscr{N}_X$ is said to be *degenerate*, or *extreme*, if κ vanishes throughout $\mathscr{N}_X$; it is called *non-degenerate* if κ has no zeros on $\mathscr{N}_X$. Thus, the Killing horizons $\mathscr{N}(X)_{\epsilon\delta}$ of (1.3.8) are non-degenerate, while both Killing horizons of X given by (1.3.10) are degenerate.

INCIDENTALLY: EXAMPLE 1.3.9 Consider the Schwarzschild metric. as extended in (1.2.10),

$$g = -(1-\frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2\,. \tag{1.3.14}$$

We have

$$d\Big(g(X,X)\Big) = d\Big(g(\partial_v,\partial_v)\Big) = -\frac{2m}{r^2}dr\,.$$

Now, $X^\flat = g(\partial_v,\cdot) = -(1-\frac{2m}{r})dv + dr$, which equals dr at the hypersurface $r=2m$. Comparing with (1.3.11) gives

$$\kappa \equiv \kappa_m := \frac{1}{4m}\,.$$

We see that the black-hole event horizon in the above extension of the Schwarzschild metric is a non-degenerate Killing horizon, with surface gravity $(4m)^{-1}$.

The same calculation for the extension based on the retarded time coordinate u of (1.2.9) proceeds identically except for various sign changes, resulting in $\kappa = -1/4m$ for the white-hole event horizon.

Note that there are no black holes with degenerate Killing horizons within the Schwarzschild family. In fact [90], there are no suitably regular, degenerate, static vacuum black holes at all.

□

In Kerr spacetimes (see Section 1.6 below) we have $\kappa = 0$ if and only if $m = a$. On the other hand, all horizons in the multi-black hole Majumdar-Papapetrou solutions of Section 1.7 are degenerate.

In what follows we will prove that κ is constant on Killing horizons under various circumstances; this is used when assigning a temperature to Killing horizons. In the proofs we will need to differentiate the defining equation

$$X^\gamma \nabla_\alpha X_\gamma \,|_{\mathscr{N}} = -\kappa X_\alpha \,. \tag{1.3.15}$$

For this some preliminary work is needed:

Let $t_{\alpha_1 \ldots \alpha_\ell}$ be any tensor field vanishing on $\mathscr{N}$. Then

$$k^\beta \nabla_\beta t_{\alpha_1 \ldots \alpha_\ell} \,|_{\mathscr{N}} = 0 \tag{1.3.16}$$

for any vector field k^β tangent to $\mathscr{N}$. Since X_β spans the space of covectors annihilating $\mathscr{N}$, (1.3.16) holds if and only if $\nabla_\beta t_{\alpha_1 \ldots \alpha_\ell}|_{\mathscr{N}}$ equals $X_\beta s_{\alpha_1 \ldots \alpha_\ell}$ for some tensor field $s_{\alpha_1 \ldots \alpha_\ell}$. Equivalently,

$$X_{[\gamma} \nabla_{\beta]} t_{\alpha_1 \ldots \alpha_\ell}|_{\mathscr{N}} = 0 \,. \tag{1.3.17}$$

This is our desired differential consequence of the vanishing of $t_{\alpha_1 \ldots \alpha_\ell}|_{\mathscr{N}}$.

We have the following:

THEOREM 1.3.10 *κ^2 is a non-zero constant on bifurcate Killing horizons.*

REMARK 1.3.11 Both (1.3.13) and the Example 1.3.9 show that κ, as defined in (1.3.11) is *not* constant on a bifurcate Killing horizon, with a sign which might change when passing from Killing horizon component to another. The reader will note that one can fiddle with the sign in (1.3.11) to obtain a constant value of κ throughout a bifurcate Killing horizon, but we will not proceed in this manner.

PROOF: We follow the argument in [167, p. 59]. Consider, quite generally, a smooth hypersurface $\mathscr{N}$ with defining function f; by definition, this means that f vanishes precisely on $\mathscr{N}$, with df different from zero on $\mathscr{N}$. Thus, on each connected component $\mathscr{N}$ of our bifurcate Killing horizon we have such a function f. Next, we claim that there exists a function h such that on $\mathscr{N}$ we have

$$X^\flat := g_{\mu\nu} X^\mu dx^\nu = h\, df \tag{1.3.18}$$

(This property is called *hypersurface orthogonality.*) Indeed, if Y is any vector tangent to $\mathscr{N}$, then $Y^\mu\partial_\mu f = 0$ on $\mathscr{N}$, since f is constant on $\mathscr{N}$. On the other hand, $Y^\mu X_\mu = 0$, because a null vector tangent to a null hypersurface $\mathscr{N}$ is orthogonal to all vectors tangent to $\mathscr{N}$. It follows that $X^\flat$ is proportional to df, which justifies the existence the function h in (1.3.18). We emphasise that we do *not* assume (1.3.18) everywhere, but only on $\mathscr{N}$.

We start by showing that (1.3.18) implies the identity

$$X_{[\mu}\nabla_\nu X_{\rho]}\,|_{\mathscr{N}} = 0\,. \tag{1.3.19}$$

Indeed, differentiating (1.3.18) we find

$$\nabla_\nu X_\rho\,|_{\mathscr{N}} = \nabla_\nu h \nabla_\rho f + h\nabla_\nu\nabla_\rho f + X_\nu Z_\rho\,, \tag{1.3.20}$$

for some vector field Z. Here, as already explained above, Z accounts for the fact that the equality (1.3.18) only holds on $\mathscr{N}$, and therefore differentiation might introduce non-zero terms in directions transverse to $\mathscr{N}$. The first term in (1.3.20) drops out under antisymmetrisation as in (1.3.19) since X is proportional to ∇f; similarly for the last one. The second term is symmetric in ρ and ν, and also gives zero under antisymmetrisation. This establishes (1.3.19). (In fact, (1.3.19) is a special case of the *Frobenius theorem*, keeping in mind that X_α is hypersurface-orthogonal.)

We continue with the identity

$$\nabla^\nu X^\rho \nabla_\nu X_\rho\,|_{\mathscr{N}} = -2\kappa^2\,. \tag{1.3.21}$$

To see this, we multiply (1.3.19) by $\nabla^\nu X^\rho$ and expand: using the symbol "$=_{\mathscr{N}}$" to denote equality on $\mathscr{N}$, we find

$$\begin{aligned}
0 \;&=_{\mathscr{N}}\; \nabla^\nu X^\rho X_\mu \nabla_\nu X_\rho + \underbrace{\nabla^\nu X^\rho X_\nu}_{\kappa X^\rho}\nabla_\rho X_\mu + \underbrace{\nabla^\nu X^\rho X_\rho}_{-\kappa X^\nu}\nabla_\mu X_\nu \\
&=_{\mathscr{N}}\; \nabla^\nu X^\rho X_\mu \nabla_\nu X_\rho + \kappa \underbrace{X^\rho\nabla_\rho X_\mu}_{\kappa X_\mu} - \kappa \underbrace{X^\nu\nabla_\mu X_\nu}_{-\kappa X_\mu} \\
&=_{\mathscr{N}}\; (\nabla^\nu X^\rho \nabla_\nu X_\rho + 2\kappa^2) X_\mu\,.
\end{aligned}$$

This proves (1.3.21) away from the set where X vanishes.

Now, recall that a bifurcate horizon is the union of four Killing horizons, which are smooth hypersurfaces on which X has no zeros, and of the bifurcation surface S, where X vanishes. So the set $\{X \neq 0\}$ is dense on a bifurcate horizon. Recall also that κ has *not* been defined on S so far, as the definition needs the condition $X \neq 0$. But we can view (1.3.21) as the definition of κ, up to sign, at points at which X vanishes. Then the calculation just given shows that the function κ^2, so-extended to S, is a smooth function on a bifurcate Killing horizon $\mathscr{N}$, with (1.3.21) holding throughout $\mathscr{N}$.

Recall (cf. (A.21.8), p. 308 below) that a Killing vector field satisfies the set of equations

$$\nabla_\alpha \nabla_\beta X_\gamma = R_{\sigma\alpha\beta\gamma} X^\sigma\,. \tag{1.3.22}$$

Differentiating (1.3.21) we obtain, using (1.3.22),

$$\nabla^\nu X_\rho \underbrace{\nabla_\sigma \nabla_\nu X_\rho}_{R_{\alpha\sigma\nu\rho}X^\alpha} |_{\mathscr{N}} = -\nabla_\sigma(\kappa^2)\,. \tag{1.3.23}$$

So the left-hand side vanishes on S. It follows that $\nabla\kappa^2$ vanishes on S. We conclude that κ^2 is constant on any connected component of S.

Contracting (1.3.23) with X^σ we further find

$$-\mathscr{L}_X\kappa^2 |_{\mathscr{N}} = -X^\sigma\nabla_\sigma(\kappa^2) |_{\mathscr{N}} = \nabla^\nu X_\rho \underbrace{R_{\alpha\sigma\nu\rho}X^\alpha X^\sigma}_{=0} = 0\,. \tag{1.3.24}$$

Hence κ^2 is constant along any the null Killing orbits threading each of the Killing horizons issued from S. Continuity implies that on each orbit the surface gravity κ^2 takes the same value as at its accumulation point at S. But we have already seen that κ^2 is constant on S. We conclude that κ^2 is constant throughout the bifurcate Killing horizon emanating from S.

It remains to show that κ^2 cannot vanish on S: This is a consequence of the fact that $X|_S = 0$, $\kappa|_S = 0$ and (1.3.21) imply $\nabla X|_S = 0$. Proposition A.21.7, p. 309 gives $X \equiv 0$, contradicting the definition of a bifurcate Killing horizon. □

Yet another class of spacetimes with constant κ (see [150], Theorem 7.1 or [271], Section 12.5) is provided by spacetimes satisfying the *dominant energy condition*: this means that $T_{\mu\nu}X^\mu Y^\nu \geq 0$ for all causal future directed vector fields X and Y. Our aim now is to prove this.

Since X_α is hypersurface-orthogonal on $\mathscr{N}$, from the "Frobenius identity" (1.3.19) we have

$$\begin{aligned} 0 \;&=_{\mathscr{N}}\; 3X_{[\beta}\nabla_\sigma X_{\gamma]} =_{\mathscr{N}} X_\beta\nabla_\sigma X_\gamma + X_\sigma\nabla_\gamma X_\beta + X_\gamma\nabla_\beta X_\sigma \\ &=_{\mathscr{N}}\; 2X_{[\beta}\nabla_{\sigma]}X_\gamma + X_\gamma\nabla_\beta X_\sigma\,. \end{aligned} \tag{1.3.25}$$

Equivalently,

$$X_{[\beta}\nabla_{\sigma]}X_\gamma |_{\mathscr{N}} = \frac{1}{2}X_\gamma\nabla_\sigma X_\beta\,. \tag{1.3.26}$$

Thus, applying the differential operator $X_{[\beta}\nabla_{\sigma]}$ to the left-hand side of (1.3.15), we find

$$\begin{aligned} X_{[\beta}\nabla_{\sigma]}(X^\gamma\nabla_\alpha X_\gamma) &= (\underbrace{X_{[\beta}\nabla_{\sigma]}X^\gamma}_{\frac{1}{2}X^\gamma\nabla_\sigma X_\beta})\nabla_\alpha X_\gamma + X^\gamma X_{[\beta}\underbrace{\nabla_{\sigma]}\nabla_\alpha X_\gamma}_{-R_{\sigma]\mu\alpha\gamma}X^\mu} \\ &= -\frac{\kappa}{2}X_\alpha\nabla_\sigma X_\beta - X_{[\beta}R_{\sigma]\mu\alpha\gamma}X^\gamma X^\mu \\ &= \kappa\, X_{[\sigma}\nabla_{\beta]}X_\alpha - X_{[\beta}R_{\sigma]\mu\alpha\gamma}X^\gamma X^\mu\,. \end{aligned} \tag{1.3.27}$$

Comparing with the corresponding derivatives of the right-hand side of (1.3.15), we conclude that

$$X_{[\beta}R_{\sigma]\mu\alpha\gamma}X^\gamma X^\mu |_{\mathscr{N}} = X_\alpha X_{[\sigma}\nabla_{\beta]}\kappa\,. \tag{1.3.28}$$

The next step is to show that

$$X_{[\beta}R_{\sigma]\mu\alpha\gamma}X^\gamma X^\mu\,|_{\mathscr{N}} = X_\alpha X_{[\beta}R_{\sigma]\gamma}X^\gamma\,. \tag{1.3.29}$$

For this, we apply $X_{[\mu}\nabla_{\nu]}$ to (1.3.19): Letting $\mathrm{S}_{\alpha\beta\gamma}$ denote a cyclic sum over $\alpha\beta\gamma$, and using (1.3.26) we obtain:

$$\begin{aligned}
0 &=_{\mathscr{N}} X_{[\mu}\nabla_{\nu]}(\mathrm{S}_{\alpha\beta\gamma}X_\alpha\nabla_\beta X_\gamma)\\
&=_{\mathscr{N}} \mathrm{S}_{\alpha\beta\gamma}((\underbrace{X_{[\mu}\nabla_{\nu]}X_\alpha}_{\frac{1}{2}X_\alpha\nabla_\nu X_\mu})\nabla_\beta X_\gamma + X_\alpha X_{[\mu}\underbrace{\nabla_{\nu]}\nabla_\beta X_\gamma}_{R_{\nu]\sigma\gamma\beta}X^\sigma})\\
&=_{\mathscr{N}} \mathrm{S}_{\alpha\beta\gamma}X_\alpha X^\sigma X_{[\mu}R_{\nu]\sigma\gamma\beta}\,.
\end{aligned} \tag{1.3.30}$$

Writing out this sum, and contracting with $g^{\alpha\mu}$, after a renaming of indices one obtains (1.3.29).

Comparing (1.3.28) and (1.3.29), since X does not vanish anywhere on a Killing horizon,

$$X_{[\alpha}\nabla_{\beta]}\kappa = -X_{[\alpha}R_{\beta]\gamma}X^\gamma\,. \tag{1.3.31}$$

We have therefore proved:

PROPOSITION 1.3.12 *Let $\mathscr{N}$ be a Killing horizon associated with a Killing vector X. If*

$$X_{[\alpha}R_{\mu]\nu}X^\nu = 0 \text{ on } \mathscr{N}\,, \tag{1.3.32}$$

then κ is constant on $\mathscr{N}$.

Let us relate (1.3.32) to the dominant energy condition, alluded to above. In vacuum the Ricci tensor vanishes, so clearly (1.3.32) is satisfied. More generally, using the Einstein equation, (1.3.32) is equivalent to

$$X_{[\alpha}T_{\mu]\nu}X^\nu = 0 \text{ on } \mathscr{N}\,. \tag{1.3.33}$$

Now, multiplying (1.3.31) by X^α and using $X^\alpha\nabla_\alpha\kappa = 0$ one finds

$$R_{\mu\nu}X^\mu X^\nu\,|_{\mathscr{N}} = 0\,, \tag{1.3.34}$$

and therefore also

$$T_{\mu\nu}X^\mu X^\nu\,|_{\mathscr{N}} = 0\,. \tag{1.3.35}$$

Assuming the dominant energy condition, this is possible if and only if (1.3.33) holds. Indeed, the condition that $T_{\mu\nu}X^\mu Y^\nu$ is positive for all causal future directed vectors implies that $-T^\mu{}_\nu X^\nu$ is causal future directed. But then $T_{\mu\nu}X^\mu X^\nu$ vanishes on $\mathscr{N}$ if and only if $T^\mu{}_\nu X^\nu$ is proportional to X^μ, which implies (1.3.33). We conclude that Proposition 1.3.12 applies, leading to:

THEOREM 1.3.13 *Let $\mathscr{N}$ be a Killing horizon and suppose that the energy-momentum tensor satisfies the dominant energy condition,*

$$T_{\mu\nu}Z^\nu Y^\mu = 0 \text{ for all causal future directed } Z \text{ and } Y\,. \tag{1.3.36}$$

Then κ is constant on $\mathscr{N}$.

We conclude this section with the following result, originally proved by Rácz and Wald [239] in spacetime dimension $n+1=4$:

THEOREM 1.3.14 *Let $\mathscr{N}$ be a Killing horizon associated with a Killing vector field X in an $(n+1)$-dimensional spacetime. Then the surface gravity κ is constant on $\mathscr{N}$ if and only if the exterior derivative of the twist form field ω is zero on the horizon, i.e.*

$$\nabla_{[\alpha_0}\omega_{\alpha_1\ldots\alpha_{n-2}]}|_{\mathscr{N}}=0\,,$$

where ω is defined as $\omega_{\alpha_1\ldots\alpha_{n-2}}=\epsilon_{\alpha_1\ldots\alpha_{n-2}\beta\gamma\delta}X^\beta\nabla^\gamma X^\delta$.

PROOF: Recall (1.3.31):

$$X_{[\alpha}\nabla_{\beta]}\kappa\,|_{\mathscr{N}}=-X_{[\alpha}R_{\beta]\gamma}X^\gamma\,. \tag{1.3.37}$$

On the other hand, letting $\delta^{\alpha\beta}_{\gamma\delta}=\delta^\alpha_{[\gamma}\delta^\beta_{\delta]}$ we have

$$\begin{aligned}
&\epsilon^{\alpha\beta\gamma\delta_1\ldots\delta_{n-2}}\nabla_\gamma\omega_{\delta_1\ldots\delta_{n-2}}\\
&= \epsilon^{\alpha\beta\gamma\delta_1\ldots\delta_{n-2}}\nabla_\gamma(\epsilon_{\delta_1\ldots\delta_{n-2}\mu\nu\rho}X^\mu\nabla^\nu X^\rho)\\
&= (-1)^{n-2}\epsilon^{\delta_1\ldots\delta_{n-2}\alpha\beta\gamma}\epsilon_{\delta_1\ldots\delta_{n-2}\mu\nu\rho}\nabla_\gamma(X^\mu\nabla^\nu X^\rho)\\
&= \underbrace{2(-1)^{n-1}(n-2)!}_{=:c_n}(\delta^\alpha_\mu\delta^{\beta\gamma}_{\nu\rho}+\delta^\beta_\mu\delta^{\gamma\alpha}_{\nu\rho}+\delta^\gamma_\mu\delta^{\alpha\beta}_{\nu\rho})\nabla_\gamma(X^\mu\nabla^\nu X^\rho)\\
&= c_n\Big(\nabla_\gamma(X^\alpha\nabla^\beta X^\gamma)+\nabla_\gamma(X^\beta\nabla^\gamma X^\alpha)+\nabla_\gamma(X^\gamma\nabla^\alpha X^\beta)\Big)\\
&= c_n(\underbrace{\nabla_\gamma X^\alpha\nabla^\beta X^\gamma}_{=:(*)}+X^\alpha\underbrace{\nabla_\gamma\nabla^\beta X^\gamma}_{=R_{\sigma\gamma}{}^{\beta\gamma}X^\sigma}+\underbrace{\nabla_\gamma X^\beta\nabla^\gamma X^\alpha}_{=-\nabla_\gamma X^\alpha\nabla^\beta X^\gamma\text{, cancels out }(*)}\\
&\quad +X^\beta\underbrace{\nabla_\gamma\nabla^\gamma X^\alpha}_{=R_{\sigma\gamma}{}^{\gamma\alpha}X^\sigma}+\underbrace{\nabla_\gamma X^\gamma}_{=0}\nabla^\alpha X^\beta+\underbrace{X^\gamma\nabla_\gamma\nabla^\alpha X^\beta}_{=X^\gamma R_{\sigma\gamma}{}^{\alpha\beta}X^\sigma=0})\\
&= 2c_nX^{[\alpha}R^{\beta]}{}_\sigma X^\sigma\,.
\end{aligned} \tag{1.3.38}$$

Comparing with (1.3.37), we find

$$X_{[\alpha}\nabla_{\beta]}\kappa\,|_{\mathscr{N}}=-\frac{1}{2c_n}\epsilon_{\alpha\beta}{}^{\gamma\delta_1\ldots\delta_{n-2}}\nabla_\gamma\omega_{\delta_1\ldots\delta_{n-2}}\,, \tag{1.3.39}$$

from which the theorem follows. □

A vector field X is said to be *hypersurface orthogonal*, if ω vanishes, compare (1.3.2). Recall that X is static if it is timelike (at least at large distances) and hypersurface orthogonal. It thus follows from Theorem 1.3.14 that the surface gravity is always constant on Killing horizons associated with *static* Killing vectors, regardless of field equations.

It is known that the twist-form vanishes for stationary and axisymmetric electro-vacuum spacetimes [177, 230]; we again infer that κ is constant for such spacetimes; of course this follows also from Theorem 1.3.13.

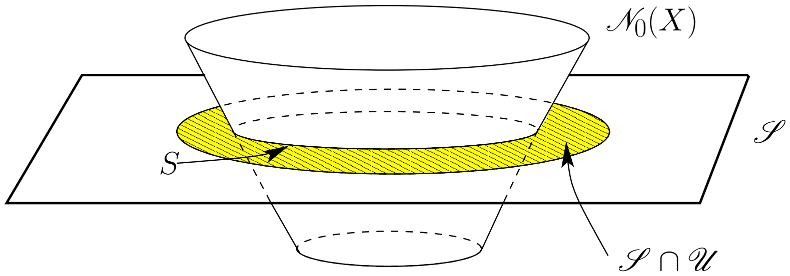

Figure 1.3.2: A spacelike hypersurface $\mathscr{S}$ intersecting a Killing horizon $\mathscr{N}_0(X)$ in a compact cross-section S.

1.3.4 The orbit-space geometry near Killing horizons

Consider a spacetime $(\mathscr{M}, \mathbf{g})$ with a Killing vector field X. On any set $\mathscr{U}$ on which X is timelike we can introduce coordinates in which $X = \partial_t$, and the metric may be written as

$$\mathbf{g} = -V^2(dt + \theta_i dx^i)^2 + g_{ij}dx^i dx^j \,, \quad \partial_t V = \partial_t \theta_i = \partial_t g_{ij} = 0 \,. \qquad (1.3.40)$$

where $g = g_{ij}dx^i dx^j$ has Riemannian signature. The metric g is often referred to as the *orbit-space metric*.

Let $\mathscr{S}$ be a spacelike hypersurface in $\mathscr{M}$; then (1.3.40) defines a Riemannian metric g on $\mathscr{S} \cap \mathscr{U}$. Assume that X is timelike on a one-sided neighborhood $\mathscr{U}$ of a Killing horizon $\mathscr{N}_0(X)$, and suppose that $\mathscr{S} \cap \mathscr{U}$ has a boundary component S which forms a compact cross-section of $\mathscr{N}_0(X)$, see Figure 1.3.2. The vanishing, or not, of the surface gravity has a deep impact on the geometry of g near $\mathscr{N}_0(X)$ [64]:

1. Every differentiable such S, included in a C^2 *degenerate* Killing horizon $\mathscr{N}_0(X)$, corresponds to a *complete* asymptotic end of $(\mathscr{S} \cap \mathscr{U}, g)$. See Figure 1.3.3.[8]

   This remains valid for stationary and axi-symmetric four-dimensional configurations without the hypothesis that X is timelike near the horizon [86].

2. Every such S included in a smooth Killing horizon $\mathscr{N}_0(X)$ on which

$$\kappa > 0 \,,$$

   corresponds to a totally geodesic boundary of $(\mathscr{S} \cap \mathscr{U}, g)$, with g being smooth up–to–boundary at S. Moreover

   (a) a doubling of $(\overline{\mathscr{S} \cap \mathscr{U}}, g)$ across S leads to a smooth metric on the doubled manifold,

[8]We are grateful to C. Williams for providing the figure.

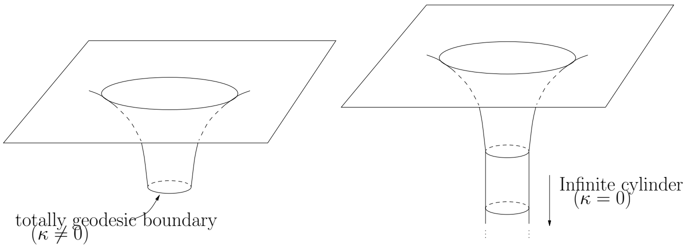

Figure 1.3.3: The general features of the geometry of the orbit-space metric on a spacelike hypersurface intersecting a non-degenerate (left) and degenerate (right) Killing horizon, near the intersection, visualized by a co-dimension one embedding in Euclidean space.

(b) with $\sqrt{-\mathbf{g}(X,X)}$ extending smoothly to $-\sqrt{-\mathbf{g}(X,X)}$ across S.

In the Majumdar-Papapetrou solutions of Section 1.7, the orbit-space metric g as in (1.3.40) asymptotes to the usual metric on a round cylinder as the event horizon is approached. One is therefore tempted to think of degenerate event horizons as corresponding to *asymptotically cylindrical ends* of $(\mathscr{S}, g)$.

1.3.5 Near-horizon geometry

A key feature of black-hole geometries is the existence of event horizons, which are null hypersurfaces. By standard causality theory, any null achronal hypersurfaces $\mathscr{H}$ is the union of Lipschitz topological hypersurfaces. Furthermore (cf. Appendix A.23, p. 316), through every point $p \in \mathscr{H}$ there is a future inextendible null geodesic entirely contained in $\mathscr{H}$ (though it may leave $\mathscr{H}$ when followed to the past of p). Such geodesics are called *generators*.

A useful tool to study geometry near *smooth* null hypersurfaces, is provided by the *null Gaussian coordinates* of Isenberg and Moncrief [210]. (It should be kept in mind that there exist null hypersurfaces which are not smooth, in fact examples exist with horizons which are nowhere C^1 [55].)

PROPOSITION 1.3.15 ([210]) *Near a smooth null hypersurface $\mathscr{H}$ one can introduce* Gaussian null coordinates*, in which the spacetime metric g takes the form*

$$g = x\varphi dv^2 + 2dvdx + 2xh_a dx^a dv + h_{ab}dx^a dx^b \,, \tag{1.3.41}$$

with $\mathscr{H}$ given by the equation $\{x = 0\}$.

PROOF: Let $S \subset \mathscr{H}$ be any $(n-1)$–dimensional submanifold of $\mathscr{H}$, transverse to the null generators of $\mathscr{H}$. Let x^a be any local coordinate system on S, and let $\ell|_S$ be any field of null vectors, defined on S, tangent to the generators of $\mathscr{H}$. Solving the equation $\nabla_\ell \ell = 0$, with initial values $\ell|_S$ on S, one obtains a null vector field ℓ defined on a $\mathscr{H}$–neighborhood $\mathscr{V} \subset \mathscr{H}$ of S, tangent to the generators of $\mathscr{H}$. One can extend x^a to $\mathscr{V}$ by solving the equation $\ell(x^a) = 0$. The function $v|_{\mathscr{H}}$ is defined by solving the equation $\ell(v) = 1$ with initial value $v|_S = 0$. Passing to a subset of $\mathscr{V}$ if necessary, this defines a global coordinate system (v, x^a) on $\mathscr{V}$. By construction we have $\ell = \partial_v$ on $\mathscr{V}$, in particular $g_{vv} = 0$ on $\mathscr{V}$. Further, ℓ is normal to $\mathscr{H}$ because $\mathscr{H}$ is a null surface, which implies $g_{va} = 0$ on $\mathscr{V}$.

Let, next, $\bar\ell|_{\mathscr{V}}$ be a field of null vectors on $\mathscr{V}$ defined uniquely by the conditions

$$g(\bar\ell|_{\mathscr{V}}, \ell) = 1\,, \quad g(\bar\ell|_{\mathscr{V}}, \partial_A) = 0\,. \tag{1.3.42}$$

The first equation implies that $\bar\ell|_{\mathscr{V}}$ is everywhere transverse to $\mathscr{V}$. Then we define $\bar\ell$ in a spacetime neighborhood $\mathscr{U} \subset \mathscr{M}$ of $\mathscr{V}$ by solving the geodesic equation $\nabla_{\bar\ell}\bar\ell = 0$ with initial value $\bar\ell|_{\mathscr{V}}$ at $\mathscr{V}$. The coordinates (v, x^a) are extended to $\mathscr{U}$ by solving the equations $\bar\ell(v) = \bar\ell(x^a) = 0$, and the coordinate x is defined by solving the equation $\bar\ell(x) = 1$, with initial value $x = 0$ at $\mathscr{V}$. Passing to a subset of $\mathscr{U}$ if necessary, this defines a global coordinate system (v, x, x^a) on $\mathscr{U}$.

By construction we have

$$\bar\ell = \partial_x\,, \tag{1.3.43}$$

hence ∂_x is a null, geodesic, vector field on $\mathscr{U}$. In particular

$$g_{xx} \equiv g(\partial_x, \partial_x) = 0\,.$$

Let $(z^A) = (x, x^a)$, and note that

$$\begin{aligned}
\bar\ell\Big(g(\bar\ell, \partial_A)\Big) &= g(\bar\ell, \nabla_{\bar\ell}\partial_A) = g(\bar\ell, \nabla_{\partial_x}\partial_A) = g(\bar\ell, \nabla_{\partial_A}\partial_x)\\
&= g(\bar\ell, \nabla_{\partial_A}\bar\ell) = \frac{1}{2}\partial_A\Big(g(\bar\ell, \bar\ell)\Big) = 0\,.
\end{aligned}$$

This shows that the components g_{xA} of the metric are x–independent. On S we have $g_{xv} = 1$ and $g_{xa} = 0$ by (1.3.42), which finishes the proof. □

INCIDENTALLY: EXAMPLE 1.3.17 An example of the coordinate system above is obtained by taking $\mathscr{H}$ to be the light-cone of the origin in $(n+1)$–dimensional Minkowski spacetime, with $x = r - t$, $y = (t + r)/2$, then the Minkowski metric η takes the form

$$\eta = -dt^2 + dr^2 + r^2 d\Omega^2 = 2dx\,dy + \frac{(x+2y)^2}{4} d\Omega^2\,.$$

□

EXAMPLE 1.3.18 The Eddington-Finkelstein coordinates, which bring the Schwarzschild metric to the form (1.2.10),

$$g = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dv\,dr + r^2 d\Omega^2\,. \tag{1.3.44}$$

provide an example of null Gauss coordinates around the null hypersurface $\{r = 2m\}$.

Quite generally, metrics of the form

$$g = -F(r)dt^2 + \tfrac{dr^2}{F(r)} + \underbrace{h_{AB}dx^A dx^B}_{=:h}\,, \tag{1.3.45}$$

where F vanishes at $r = r_0$, can be extended across the null-hypersurface $\{r = r_0\}$ by introducing a new coordinate

$$v = t + f(r)\,, \text{ where } f' = \frac{1}{F}\,. \tag{1.3.46}$$

This leads to

$$g = -F dv^2 + 2dv\, dr + h\,, \tag{1.3.47}$$

which is directly in the null-Gaussian-form (1.3.41). □

Average surface gravity

A topological submanifold S of a null achronal hypersurface $\mathscr{H}$ will be called a *local section*, or simply *section*, if S meets the generators of $\mathscr{H}$ transversally; it will be called a *cross-section* if it meets all the generators precisely once.

Let S be any smooth compact cross-section of $\mathscr{H}$, the *average surface gravity* $\langle\kappa\rangle_S$ is defined as

$$\langle\kappa\rangle_S = -\frac{1}{2|S|}\int_S \varphi d\mu_h\,, \tag{1.3.48}$$

where φ is as in (1.3.41), $d\mu_h$ is the measure induced by the metric h on S, and $|S|$ is the volume of S.

We emphasise that while the notion of surface gravity was defined for Killing horizons, that of average surface gravity is defined for any sufficiently differentiable null hypersurface with compact cross-sections. Note that the requirement of compactness is crucial to guarantee finiteness of the defining integral.

Suppose, however, that $X := \partial_v$ is a Killing vector, and that $\mathscr{H}$ is an associated Killing horizon. Then φ is directly related to the surface gravity of $\mathscr{H}$: Indeed, from (1.3.41) we have

$$g(X,X) \equiv g_{vv} = x\varphi\,, \quad d(g(X,X))|_{x=0} = \varphi dx\,, \quad \nabla(g(X,X))|_{x=0} = \varphi\partial_v\,,$$

and the definition (1.3.11), p. 40, of surface gravity κ gives

$$\kappa = -\frac{1}{2}\varphi\,.$$

So if κ is constant on $\mathscr{H}$ which, as discussed in Section 1.3.3. holds in many situations of interest, we obtain

$$\langle\kappa\rangle_S = \kappa\,.$$

The near-horizon geometry equations

When $\mathscr{H}$ is a degenerate Killing horizon, the surface gravity vanishes by definition. This implies that the function φ can itself be written as xA, for some smooth function A. The vacuum Einstein equations imply (see [210, eq. (2.9)] in dimension four and [187, eq. (5.9)] in higher dimensions)

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{h}_a\mathring{h}_b - \mathring{D}_{(a}\mathring{h}_{b)} \tag{1.3.49}$$

where, using the notation of (1.3.41), $\mathring{R}_{ab}$ is the Ricci tensor of $\mathring{h}_{ab} := h_{ab}|_{x=0}$, and $\mathring{D}$ is the covariant derivative thereof, while $\mathring{h}_a := h_a|_{x=0}$. The Einstein equations also determine $\mathring{A} := A|_{x=0}$ uniquely in terms of $\mathring{h}_a$ and $\mathring{h}_{ab}$:

$$\mathring{A} = \frac{1}{2}\mathring{h}^{ab}\left(\mathring{h}_a\mathring{h}_b - \mathring{D}_a\mathring{h}_b\right) \tag{1.3.50}$$

(this equation follows again e.g. from [210, eq. (2.9)] in dimension four, and can be checked by a calculation in all higher dimensions).

The triple $(S, \mathring{h}_{ab}, \mathring{h}_a)$ is called the *near-horizon geometry.* In view of the Taylor expansions

$$h_a = \mathring{h}_a + O(x)\,, \quad h_{ab} = \mathring{h}_{ab} + O(x)\,,$$

the pair $(\mathring{h}_{ab}, \mathring{h}_a)$ together with (1.3.50) describes the leading-order behaviour of the metric near $\{x = 0\}$, which justifies the name.

Suppose that g satisfies the vacuum Einstein equations, possibly with a cosmological constant. If ∂_v is a Killing vector (equivalently, $\partial_v A = 0 = \partial_v h_a = \partial_v h_{ab}$), then the *near-horizon metric*

$$\mathring{g} = x^2\mathring{A}dv^2 + 2dvdx + 2x\mathring{h}_a dx^a dv + \mathring{h}_{ab}dx^a dx^b \tag{1.3.51}$$

is also a solution of the vacuum Einstein equations. To see this, let $\epsilon > 0$ and in the metric (1.3.41) replace the coordinates (v, x) by $(\epsilon^{-1}v, \epsilon x)$;

$$\begin{aligned} g_\epsilon &:= g|_{(v,x)\to(\epsilon^{-1}v,\epsilon x)} \\ &= \epsilon^2x^2A(\epsilon x, x^a)d(\epsilon^{-1}v)^2 + 2d(\epsilon^{-1}v)d(\epsilon x) + 2\epsilon x h_a(\epsilon x, x^a)dx^a d(\epsilon^{-1}v) \\ &\quad + h_{ab}(\epsilon x, x^a)dx^a dx^b \\ &= x^2A(\epsilon x, x^a)dv^2 + 2dvdx + 2xh_a(\epsilon x, x^a)dx^a dv + h_{ab}(\epsilon x, x^a)dx^a dx^b \\ &\quad \to_{\epsilon\to 0} \mathring{g}\,. \end{aligned} \tag{1.3.52}$$

Now, for every $\epsilon > 0$ the metric g_ϵ is in fact g written in a different coordinate system. Hence the before-last line of (1.3.52) provides a family g_ϵ of solutions of the vacuum Einstein equations depending smoothly on a parameter ϵ. Passing to the limit $\epsilon \to 0$, the conclusion readily follows.

The classification of near-horizon geometries turns out to be a key step towards a classification of degenerate black holes. We have[9] the following partial results, where either staticity is assumed without restriction on dimensions, or axial-symmetry is required in spacetime dimension four [187]:

THEOREM 1.3.19 ([90]) *Let the spacetime dimension be $n+1$, $n \geq 3$, suppose that a degenerate Killing horizon $\mathscr{N}$ has a compact cross-section, and that $\mathring{h}_a = \partial_a\lambda$ for some function λ (which is necessarily the case in vacuum static spacetimes). Then (1.3.49) implies $\mathring{h}_a \equiv 0$, so that $\mathring{h}_{ab}$ is Ricci-flat.*

THEOREM 1.3.20 ([187]) *In spacetime dimension four and in vacuum, suppose that a degenerate Killing horizon $\mathscr{N}$ has a spherical cross-section, and that $(\mathscr{M}, g)$ admits a second Killing vector field with periodic orbits. For every connected component $\mathscr{N}_0$ of $\mathscr{N}$ there exists an embedding of $\mathscr{N}_0$ into a Kerr spacetime which preserves $\mathring{h}_a$, $\mathring{h}_{ab}$ and $\mathring{A}$.*

[9] Some partial results with a non-zero cosmological constant have also been proved in [90].

In the four-dimensional static case, Theorem 1.3.19 enforces toroidal topology of cross-sections of $\mathscr{N}$, with a flat $\mathring{h}_{ab}$. This, together with the sphericity theorem [96], shows non-existence of static, degenerate, asymptotically flat, suitably regular vacuum black holes.

On the other hand, in the four-dimensional axi-symmetric case, Theorem 1.3.20 guarantees that the geometry tends to a Kerr one, up to second order errors, when the horizon is approached. This is one of the key ingredients of the proof of the uniqueness theorem for axi-symmetric, degenerate, connected, vacuum, asymptotically flat, suitably regular black holes [86].

It would be of significant interest to obtain more information about solutions of (1.3.49), in all dimensions, without any restrictive conditions. For instance, it is expected that the hypothesis of the existence of a second vector field is not necessary for Theorem 1.3.20, and it would of interest to prove, or disprove, this.

INCIDENTALLY: A partial result towards the existence of a second Killing vector has been obtained in [93], where small perturbations of the Kerr near-horizon geometry are studied. For such perturbations the problem can be reduced to a study of the linearised equations. Using a formalism introduced by Jezierski and Kaminski [165] and spherical harmonic decompositions one reduces the problem to the proof that each of the spherical harmonic modes has no kernel. This is done there analytically except for the seven lowest modes, for which numerical evidence is provided. A key step of the analysis, established in [93] without any smallness assumptions, is the proof that $\mathring{h}$ always has precisely two zeros of index one.

Some further results concerning the problem can be found in [165, 222].

As just seen, in the degenerate case the vacuum equations impose strong restrictions on the near-horizon geometry. It turns out that no such restrictions exist for non-degenerate horizons, at least in the analytic setting: Indeed, for any triple $(N, \mathring{h}_a, \mathring{h}_{ab})$, where N is a two-dimensional analytic manifold (compact or not), $\mathring{h}_a$ is an analytic one-form on N, and $\mathring{h}_{ab}$ is an analytic Riemannian metric on N, there exists a vacuum spacetime $(\mathscr{M}, g)$ with a bifurcate (and thus non-degenerate) Killing horizon, so that the metric g takes the form (1.3.41) near each Killing horizon branching out of the bifurcation surface $S \approx N$, with $\mathring{h}_{ab} = h_{ab}|_{r=0}$ and $\mathring{h}_a = h_a|_{r=0}$; in fact $\mathring{h}_{ab}$ is the metric induced by g on S. When N is the two-dimensional torus $\mathbb{T}^2$ this can be inferred from [209] as follows: using [209, Theorem (2)] with $(\phi, \beta_a, g_{ab})|_{t=0} = (0, 2\mathring{h}_a, \mathring{h}_{ab})$ one obtains a vacuum spacetime $(\mathscr{M}' = S^1 \times \mathbb{T}^2 \times (-\epsilon, \epsilon), g')$ with a compact Cauchy horizon $S^1 \times \mathbb{T}^2$ and Killing vector X tangent to the S^1 factor of $\mathscr{M}'$. One can then pass to a covering space where S^1 is replaced by $\mathbb{R}$, and use a construction of Rácz and Wald (cf. Theorem 1.7.3 below) to obtain the desired $\mathscr{M}$ containing the bifurcate horizon.

This argument generalises to any analytic $(N, \mathring{h}_a, \mathring{h}_{ab})$ without difficulties.

1.3.6 Asymptotically flat stationary metrics

There exists several ways of defining asymptotic flatness, all of them roughly equivalent in vacuum. We will adapt a Cauchy data point of view, as it ap-
•1.3.1: pears to be the least restrictive. •1.3.1 So, a spacetime $(\mathscr{M}, \mathbf{g})$ will be said to

possess an *asymptotically flat end* if $\mathscr{M}$ contains a spacelike hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R, with the following properties: there exists a constant $\alpha > 0$ such that, in local coordinates on $\mathscr{S}_{\rm ext}$ obtained from $\mathbb{R}^n \setminus B(R)$, the metric g induced by $\mathbf{g}$ on $\mathscr{S}_{\rm ext}$, and the extrinsic curvature tensor K of $\mathscr{S}_{\rm ext}$, satisfy the fall-off conditions, for some $k > 1$,

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha}) , \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}) , \tag{1.3.53}$$

where we write $f = O_k(r^\alpha)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = O(r^{\alpha-\ell}) , \quad 0 \le \ell \le k . \tag{1.3.54}$$

For simplicity we assume that the spacetime is vacuum, though similar results hold in general under appropriate conditions on matter fields, see [19, 82] and references therein. Along any spacelike hypersurface $\mathscr{S}$, a Killing vector field X of $(\mathscr{M}, \mathbf{g})$ can be decomposed as

$$X = Nn + Y ,$$

where Y is tangent to $\mathscr{S}$, and n is the unit future-directed normal to $\mathscr{S}_{\rm ext}$. The fields N and Y are called "Killing initial data", or *KID* for short. The vacuum field equations, together with the Killing equations imply the following set of equations on $\mathscr{S}$:

$$D_i Y_j + D_j Y_i = 2N K_{ij} , \tag{1.3.55}$$

$$R_{ij}(g) + K^k{}_k K_{ij} - 2K_{ik}K^k{}_j - N^{-1}(\mathscr{L}_Y K_{ij} + D_i D_j N) = 0 , \tag{1.3.56}$$

where $R_{ij}(g)$ is the Ricci tensor of g. Equations (1.3.55)-(1.3.56) will be referred to as the *vacuum KID equations*.

Under the boundary conditions (1.3.53), an analysis of these equations provides detailed information about the asymptotic behavior of (N, Y). In particular one can prove that if the asymptotic region $\mathscr{S}_{\rm ext}$ is part of initial data set $(\mathscr{S}, g, K)$ satisfying the requirements of the positive energy theorem, and if X is timelike along $\mathscr{S}_{\rm ext}$, then $(N, Y^i) \to_{r\to\infty} (A^0, A^i)$, where the A^μ's are constants satisfying $(A^0)^2 > \sum_i (A^i)^2$ [17, 82]. One can then choose adapted coordinates so that the metric can be, locally, written as

$$\mathbf{g} = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij} dx^i dx^j}_{=g} , \tag{1.3.57}$$

with

$$\partial_t V = \partial_t \theta = \partial_t g = 0 \tag{1.3.58}$$

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha}) , \quad \theta_i = O_k(r^{-\alpha}) , \quad V - 1 = O_k(r^{-\alpha}) . \tag{1.3.59}$$

As discussed in more detail in [20], in g-harmonic coordinates, and in e.g. a maximal time-slicing, the vacuum equations for $\mathbf{g}$ form a quasi-linear elliptic system with diagonal principal part, with principal symbol identical to that of the scalar Laplace operator. Methods known in principle show that, in this "gauge", all metric functions have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r. In the new coordinates we can in fact take

$$\alpha = n - 2\,. \tag{1.3.60}$$

By inspection of the equations one can further infer that the leading order corrections in the metric can be written in the Schwarzschild form (1.2.50).

Solutions without $\ln r$ terms are of special interest, because the associated spacetimes have smooth conformal completion at infinity. In even spacetime dimension initial data sets containing such asymptotic regions, when close enough to Minkowskian data, lead to asymptotically simple spacetimes [9, 52, 120]. It has been shown by Beig and Simon that logarithmic terms can always be gotten rid of by a change of coordinates in space dimension three when the mass is non-zero [21, 256]. This has been generalised in [20] to all stationary metrics in even space-dimension $n \geq 6$, and to static metrics with non-vanishing mass in $n = 5$.

1.3.7 Domains of outer communications, event horizons

A key notion in the theory of stationary asymptotically flat black holes is that of the *domain of outer communications*, defined as follows: For $t \in \mathbb{R}$ let $\phi_t[X] : \mathscr{M} \to \mathscr{M}$ denote the one-parameter group of diffeomorphisms generated by a Killing vector field X; we will write ϕ_t for $\phi_t[X]$ whenever ambiguities are unlikely to occur. Let $\mathscr{S}_{\rm ext}$ be as in Section 1.3.6. The exterior region $\mathscr{M}_{\rm ext}$ and the *domain of outer communications* $\langle\langle \mathscr{M} \rangle\rangle$ are then defined as[10]

$$\mathscr{M}_{\rm ext} := \cup_t \phi_t(\mathscr{S}_{\rm ext})\,, \qquad \langle\langle \mathscr{M} \rangle\rangle = I^+(\mathscr{M}_{\rm ext}) \cap I^-(\mathscr{M}_{\rm ext})\,. \tag{1.3.61}$$

The *black hole region* $\mathscr{B}$ and the *black hole event horizon* $\mathscr{H}^+$ are defined as

$$\mathscr{B} = \mathscr{M} \setminus I^-(\mathscr{M}_{\rm ext})\,, \quad \mathscr{H}^+ = \partial \mathscr{B}\,.$$

The *white hole region* $\mathscr{W}$ and the *white hole event horizon* $\mathscr{H}^-$ are defined as above after changing time orientation:

$$\mathscr{W} = \mathscr{M} \setminus I^+(\mathscr{M}_{\rm ext})\,, \quad \mathscr{H}^- = \partial \mathscr{W}\,.$$

It follows that the boundaries of $\langle\langle \mathscr{M} \rangle\rangle$ are included in the event horizons. We set

$$\mathscr{E}^\pm = \partial \langle\langle \mathscr{M} \rangle\rangle \cap I^\pm(\mathscr{M}_{\rm ext})\,, \qquad \mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^-\,. \tag{1.3.62}$$

There is considerable freedom in choosing the asymptotic region $\mathscr{S}_{\rm ext}$. However, it is not too difficult to show that $I^\pm(\mathscr{M}_{\rm ext})$, and hence $\langle\langle \mathscr{M} \rangle\rangle$, $\mathscr{H}^\pm$ and $\mathscr{E}^\pm$, are independent of the choice of $\mathscr{S}_{\rm ext}$ as long as the associated $\mathscr{M}_{\rm ext}$'s overlap.

[10] Recall that $I^-(\Omega)$, respectively $J^-(\Omega)$, is the set covered by past-directed timelike, respectively causal, curves originating from Ω, while $\dot I^-$ denotes the boundary of I^-, etc. The sets I^+, etc., are defined as I^-, etc., after changing time-orientation.

Our definitions of domain of outer communications, black hole region, etc., have been tailored to asymptotically flat stationary spacetimes. The definitions carry over *verbatim* to spacetimes with different asymptotics when a preferred region $\mathscr{M}_{\rm ext}$ is present, as e.g. for asymptotically anti-de Sitter spacetimes. As such, an approach based on conformal completions, and presented in Section 5.1, p. 201, is often used in the literature. It is applicable to more general spacetimes, as it does not require stationarity. In the stationary case the definitions just given are equivalent to the "conformal ones" in all standard examples, and avoid the various and irrelevant problems introduced by the conformal completions and discussed in Section 5.1.

In many examples presented in this work, *Killing horizons* coincide with *event horizons* and, in fact, there exist general statements to this effect in the literature.

1.4 Extensions

Before continuing our studies of various aspects of black-hole spacetimes, it is convenient to discuss systematically the notion of extensions of Lorentzian manifolds, and of their properties. Our presentation follows closely that of [71].

Let $k \in \mathbb{R} \cup \{\infty\} \cup \{\omega\}$. The $(n+1)$–dimensional spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$ is said to be *a C^k–extension* of an $(n+1)$–dimensional spacetime $(\mathscr{M}, g)$ if there exists a C^k–immersion $\psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ such that $\psi^*\widetilde{g} = g$, and such that $\psi(\mathscr{M}) \neq \widetilde{\mathscr{M}}$. A spacetime $(\mathscr{M}, g)$ is said to be C^k*-maximal*, or C^k*-inextendible*, if no C^k–extensions of $(\mathscr{M}, g)$ exist.

1.4.1 Distinct extensions

We start by noting that maximal analytic extensions of manifolds are not unique. The simplest examples have already been discussed in Remark 1.2.9: remove a subset Ω from a maximally extended manifold $\mathscr{M}$ so that $\mathscr{M} \setminus \Omega$ is not simply connected, and pass to the universal cover; extend maximally the spacetime so obtained, if further needed. This provides many distinct maximal extensions. One is tempted to believe that such constructions can be used to classify all maximal analytic extensions, but this remains to be seen.

One can likewise ask the question, whether uniqueness holds in the class of globally hyperbolic extensions. The following variation of the last construction gives a negative answer, when "inextendible" is meant as "inextendible within the class of globally hyperbolic manifolds": Let $(\mathscr{M}, g)$ be a simply connected, analytic, globally hyperbolic spacetime and let $(\widehat{\mathscr{M}}, \widehat{g})$ be an inextendible, simply connected, analytic, globally hyperbolic extension of $(\mathscr{M}, g)$. Let $\mathscr{S}$ be a Cauchy surface in $\widehat{\mathscr{M}}$, and remove from $\mathscr{S} \setminus \mathscr{M}$ a closed subset Ω so that $\mathscr{S} \setminus \Omega$ is not simply connected. Let $\widetilde{\mathscr{S}}$ be a maximal analytic extension of the universal covering space of $\mathscr{S} \setminus \Omega$, with the obvious Cauchy data inherited from $\mathscr{S}$, and let $(\widetilde{\mathscr{M}}, \widetilde{g})$ be the maximal globally hyperbolic development thereof. Then $(\widetilde{\mathscr{M}}, \widetilde{g})$ is a globally hyperbolic analytic extension of $(\mathscr{M}, g)$ which is maximal in the class of globally hyperbolic manifolds, and distinct from $(\widehat{\mathscr{M}}, \widehat{g})$.

The examples just discussed exhibit the following undesirable feature: existence of maximally extended geodesics of affine length near which the spacetime is locally extendible in the sense of [238]. This *local extendibility* pathology can be avoided by exploiting certain symmetries of the extensions, as follows:

Consider any spacetime $(\widehat{\mathscr{M}},\widehat{g})$, and let $\mathscr{M}$ be a proper open subset of $\widehat{\mathscr{M}}$ with the metric g obtained by restriction. Thus $(\widehat{\mathscr{M}},\widehat{g})$ is an extension of $(\mathscr{M},g)$. Suppose that there exists a non-trivial isometry Ψ of $(\widehat{\mathscr{M}},\widehat{g})$ satisfying: a) Ψ has no fixed points; b) $\Psi(\mathscr{M})\cap\mathscr{M}=\emptyset$; and c) Ψ^2 is the identity map. Then, by a) and c), $\widehat{\mathscr{M}}/\Psi$ equipped with the obvious metric (still denoted by g) is a Lorentzian manifold. Furthermore, by b), $\mathscr{M}$ embeds diffeomorphically into $\widehat{\mathscr{M}}/\Psi$ in the obvious way, and therefore $\widehat{\mathscr{M}}/\Psi$ also is an extension of $\mathscr{M}$, distinct from $(\widehat{\mathscr{M}},\widehat{g})$.

It follows from the results in [220] that $(\widehat{\mathscr{M}}/\Psi,g)$ is analytic if $(\mathscr{M},g)$ was (compare [62, Appendix A]).

Keeping in mind that a spacetime must be time-oriented by definition, $\widehat{\mathscr{M}}/\Psi$ will be a spacetime if and only if Ψ preserves time-orientation. If $\widehat{\mathscr{M}}$ is simply connected, then $\pi_1(\widehat{\mathscr{M}}/\Psi)=\mathbb{Z}_2$.

Example 1.4.1 As a definite example of this construction, denote by $(\widehat{\mathscr{M}},\widehat{g})$ the Kruskal-Szekeres extension of the Schwarzschild spacetime $(\mathscr{M},g)$; by the latter we mean a connected component of the set $\{r>2m\}$ within $\widehat{\mathscr{M}}$. Let (T,X) be the global coordinates on $\widehat{\mathscr{M}}$ as defined in (1.2.28). Let $\mathring{\Psi}:S^2\to S^2$ be the antipodal map. For $p\in S^2$ consider the four isometries $\Psi_{\pm\pm}$ of the Kruskal-Szekeres spacetime defined as

$$\Psi_{\pm\pm}(T,X,p)=(\,\pm T,\pm X,\mathring{\Psi}(p))\,.$$

Set $\mathscr{M}_{\pm\pm}:=\widehat{\mathscr{M}}/\Psi_{\pm\pm}$. Since Ψ_{++} is the identity, $\mathscr{M}_{++}=\widehat{\mathscr{M}}$ is the Kruskal-Szekeres manifold, so nothing of interest here. Next, both manifolds $\mathscr{M}_{-\pm}$ are smooth maximal analytic Lorentzian extensions of $(\mathscr{M},g)$, but are not spacetimes because the maps $\Psi_{-\pm}$ do not preserve time-orientation. However, $\mathscr{M}_{+-}$ provides a maximal globally hyperbolic analytic extension of the Schwarzschild manifold distinct from $\widehat{\mathscr{M}}$. This is the "$\mathbb{RP}^3$ geon" discussed in [118]. □

1.4.2 Inextendibility

A *scalar invariant* is a function which can be calculated using the geometric objects at hand and which is invariant under coordinate transformations.

For instance, a function α_g which can be calculated in local coordinates from the metric g and its derivatives will be a scalar invariant if, for any local diffeomorphism ψ we have

$$\alpha_g(p)=\alpha_{\psi^*g}(\psi^{-1}(p))\,. \tag{1.4.1}$$

In the case of the scalar invariant $g(X,X)$ calculated using a metric g and a Killing vector X, the invariance property (1.4.1) is replaced by

$$\alpha_{g,X}(p)=\alpha_{\psi^*g,(\psi^{-1})^*X}(\psi^{-1}(p))\,. \tag{1.4.2}$$

A scalar invariant f on $(\mathscr{M}, g)$ will be called a C^k–*compatibility scalar* if f satisfies the following property: For every C^k–extension $(\widetilde{\mathscr{M}}, \widetilde{g})$ of $(\mathscr{M}, g)$ and for any bounded timelike geodesic segment γ in $\mathscr{M}$ such that $\psi(\gamma)$ accumulates at the boundary $\partial(\psi(\mathscr{M}))$ (where ψ is the immersion map $\psi : \mathscr{M} \to \widetilde{\mathscr{M}}$), the function f is bounded along γ.

Any constant function is a compatibility scalar in this terminology, albeit not very useful in practice. An example of a useful C^2–compatibility scalar is the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. Another example is provided by the norm $g(X, X)$ of a Killing vector X of g:[11]

THEOREM 1.4.2 *Let X be a Killing vector field on $(\mathscr{M}, g)$ and suppose that there exists a curve γ on which $g(X, X)$ is unbounded. Then there exists no extension $(\widehat{\mathscr{M}}, \hat{g})$ of $(\mathscr{M}, g)$, with a C^2 metric $\hat{g}$, in which the curve γ acquires an end point.*

PROOF: Suppose that γ, when viewed as a curve in $\widehat{\mathscr{M}}$, acquires an end point $p \in \widehat{\mathscr{M}}$. The linear system of equations (A.21.11), p. 309 below, satisfied by the Killing vector and its derivatives along γ, shows that $X \circ \gamma$ extends by continuity to p. This implies that $g(X, X)$ remains bounded along γ, contradicting our hypothesis. $\square$

The next inextendibility criterion from [71] is often useful:

PROPOSITION 1.4.3 *Suppose that every timelike geodesic γ in $(\mathscr{M}, g)$ is either complete, or some C^k–compatibility scalar is unbounded on γ. Then $(\mathscr{M}, g)$ is C^k-inextendible.*

PROOF: Suppose that there exists a C^k–extension $(\widetilde{\mathscr{M}}, \widetilde{g})$ of $(\mathscr{M}, g)$, with immersion $\psi : \mathscr{M} \to \widetilde{\mathscr{M}}$. We identify $\mathscr{M}$ with its image $\psi(\mathscr{M})$ in $\widetilde{\mathscr{M}}$.

Let $p \in \partial\mathscr{M}$ and let $\mathscr{O}$ be a globally hyperbolic neighborhood of p. Let $q_n \in \mathscr{M}$ be a sequence of points approaching p, thus $q_n \in \mathscr{O}$ for n large enough. Suppose, first, that there exists n such that $q_n \in I^+(p) \cup I^-(p)$. By global hyperbolicity of $\mathscr{O}$ there exists a timelike geodesic segment γ from q_n to p. Then the part of γ which lies within $\mathscr{M}$ is inextendible and has finite affine length. Furthermore every C^k–compatibility scalar is bounded on γ. But there are no such geodesics through q_n by hypothesis. We conclude that

$$(I^+(p) \cup I^-(p)) \cap \mathscr{M} = \emptyset \,. \tag{1.4.3}$$

Let $q \in (I^+(p) \cup I^-(p)) \cap \mathscr{O}$, thus $q \notin \mathscr{M}$ by (1.4.3). Since $I^+(q) \cup I^-(q)$ is open, and $p \in I^+(q) \cup I^-(q)$, we have $q_n \in I^+(q) \cup I^-(q)$ for all n sufficiently large, say $n \geq n_0$. Let γ be a timelike geodesic segment from q_{n_0} to q. Since q is not in $\mathscr{M}$, the part of γ that lies within $\mathscr{M}$ is inextendible within $\mathscr{M}$ and has finite affine length, with all C^k–compatibility scalars bounded. This is again incompatible with our hypotheses, and the result is established. $\square$

[11]This inextendibility criterion has been introduced in [23] (see the second part of Proposition 5, p. 139 there).

1.4.3 Uniqueness of a class of extensions

In this section we address the question of uniqueness of analytic extensions.

We start with some terminology. A maximally extended geodesic ray $\gamma : [0, s^+) \to \mathscr{M}$ will be called *s–complete* if $s_+ = \infty$ *unless* there exists some polynomial scalar invariant α such that

$$\limsup_{s\to s_+} |\alpha(\gamma(s))| = \infty\,.$$

A similar definition applies to maximally extended geodesics $\gamma : (s_-, s^+) \to \mathscr{M}$, with some polynomial scalar invariant (not necessarily the same) unbounded in the incomplete direction, if any. Here, by a *polynomial scalar invariant* we mean a scalar function which is a polynomial in the metric, its inverse, the Riemann tensor and its derivatives. It should be clear how to include in this notion some other objects of interest, such as the norm $g(X, X)$ of a Killing vector X, or of a Yano-Killing tensor, etc. But care should be taken not to take scalars such as $\ln(R_{ijkl}R^{ijkl})$ which could blow up even though the geometry remains regular; this is why we restrict attention to polynomials.

A Lorentzian manifold $(\mathscr{M}, g)$ will be said to be *s–complete* if every maximally extended geodesic is s–complete. The notions of *timelike s–completeness*, or *causal s–completeness* are defined similarly, by specifying the causal type of the geodesics in the definition above.

We have the following version of [172, Theorem 6.3, p. 255] (compare also the *Remark* on p. 256 there), where geodesic completeness is weakened to *timelike s–completeness*:

Theorem 1.4.4 *Let $(\mathscr{M}, g)$, $(\mathscr{M}', g')$ be analytic Lorentzian manifolds of dimension $n + 1$, $n \geq 1$, with $\mathscr{M}$ connected and simply connected, and $\mathscr{M}'$ timelike s–complete. Then every isometric immersion $f_U : U \subset \mathscr{M} \hookrightarrow \mathscr{M}'$, where U is an open subset of $\mathscr{M}$, extends uniquely to an isometric immersion $f : \mathscr{M} \hookrightarrow \mathscr{M}'$.*

We start by noting two preliminary lemmas, which are proved as in [172] by replacing "*affine mappings*" there by "*isometric immersions*":

Lemma 1.4.5 *(* [172, Lemma 1, p. 252]) *Let $\mathscr{M}$, $\mathscr{M}'$ be analytic manifolds, with $\mathscr{M}$ connected. Let f, g be analytic mappings $\mathscr{M} \to \mathscr{M}'$. If f and g coincide on a nonempty open subset of $\mathscr{M}$, then they coincide everywhere.* □

Lemma 1.4.6 *(* [172, Lemma 4, p. 254]) *Let $(\mathscr{M}, g)$ and $(\mathscr{M}', g')$ be pseudo-Riemannian manifolds of same dimension, with $\mathscr{M}$ connected, and let f and g be isometric immersions of $\mathscr{M}$ into $\mathscr{M}'$. If there exists some point $x \in \mathscr{M}$ such that $f(x) = g(x)$ and $f_*(X) = g_*(X)$ for every vector X of $T_x\mathscr{M}$, then $f = g$ on $\mathscr{M}$.* □

Before passing to the proof Theorem 1.4.4, we note a simple Corollary:

Corollary 1.4.7 *Let $(\mathscr{M}, g)$, $(\mathscr{M}', g')$ be two connected, simply connected, s-complete analytic Lorentzian extensions of $(U, \mathring{g})$. Then there exists an isometric diffeomorphism $f : \mathscr{M} \to \mathscr{M}'$.*

Proof of Corollary 1.4.7: Viewing U as a subset of $\mathscr{M}$, Theorem 1.4.4 provides an isometric immersion $f : \mathscr{M} \hookrightarrow \mathscr{M}'$ such that $f|_U = \mathrm{id}_U$. Viewing U as a subset of $\mathscr{M}'$, Theorem 1.4.4 provides an isometric immersion $f' : \mathscr{M}' \hookrightarrow \mathscr{M}$ such that $f|_U = \mathrm{id}$. Then $f \circ f'$ is an isometry of $(\mathscr{M}', g')$ satisfying $(f \circ f')|_U = \mathrm{id}_U$, hence $f \circ f' = \mathrm{id}_{\mathscr{M}'}$ by Lemma 1.4.6. Similarly $f' \circ f = \mathrm{id}_{\mathscr{M}}$, as desired. □

We can turn our attention now to the proof of Theorem 1.4.4:

Proof of Theorem 1.4.4: Similarly to the proof of Theorem 6.1 in [172], we define an *analytic continuation of f_U along a continuous path* $c : [0,1] \to \mathscr{M}$ to be a set of mappings f_s, $0 \le s \le 1$, together with a family of open subsets U_s, $0 \le s \le 1$, satisfying the properties:

- $f_0 = f_U$ on $U_0 = U$;
- for every $s \in [0,1]$, U_s is a neighborhood of the point $c(s)$ of the path c, and f_s is an isometric immersion $f_s : U_s \subset \mathscr{M} \hookrightarrow \mathscr{M}'$;
- for every $s \in [0,1]$, there exists a number $\delta_s > 0$ such that for all $s' \in [0,1]$, $(|s' - s| < \delta_s) \Rightarrow (c(s') \in U_s$ and $f_{s'} = f_s$ in a neighborhood of $c(s'))$.

We need to prove that, under the hypothesis of s–completeness, such an analytic continuation does exist along any curve c. The argument is simplest for timelike curves, so let us first assume that c is timelike. To do so, we consider the set:

$$A := \{s \in [0,1] \mid \text{an analytic continuation exists along } c \text{ on } [0,s]\} \qquad (1.4.4)$$

A is nonempty, as it contains a neighborhood of 0. Hence $\bar{s} := \sup A$ exists and is positive. We need to show that in fact, $\bar{s} = 1$ and can be reached. Assume that this is not the case. Let W be a normal convex neighborhood of $c(\bar{s})$ such that every point x in W has a normal neighborhood containing W. (Such a W exists from Theorem 8.7, chapter III of [172].) We can choose $s_1 < \bar{s}$ such that $c(s_1) \in W$, and we let V be a normal neighborhood of $c(s_1)$ containing W. Since $s_1 \in A$, f_{s_1} is well defined, and is an isometric immersion of a neighborhood of $c(s_1)$ into $\mathscr{M}'$; we will extend it to $V \cap I^{\pm}(c(s_1))$. To do so, we know that $\exp : V^* \to V$ is a diffeomorphism, where V^* is a neighborhood of 0 in $T_{c(s_1)}\mathscr{M}$, hence, in particular, for $y \in V \cap I^{\pm}(c(s_1))$, there exists a unique $X \in V^*$ such that $y = \exp X$. Define $X' := f_{s_1 *}X$. Then X' is a vector tangent to $\mathscr{M}'$ at the point $f_{s_1}(c(s_1))$. Since y is in the timelike cone of $c(\bar{s})$, X is timelike, and so is X', as f_{s_1} is isometric. We now need to prove the following:

Lemma 1.4.8 *The geodesic $s \mapsto \exp(sX')$ of $\mathscr{M}'$ is well defined for $0 \le s \le 1$.*

Proof: Let

$$s^* := \sup\{s \in [0,1] \mid \exp(s'X') \text{ exists } \forall s' \in [0,s]\}. \qquad (1.4.5)$$

First, such a s^* exists, is positive, and we notice that if $s^* < 1$, then it is not reached. We wish to show that $s^* = 1$ and is reached. Hence, it suffices to show that "s^* is not reached" leads to a contradiction. Indeed, in such a case the timelike geodesic $s \mapsto \exp(sX')$ ends at finite affine parameter, thus, there exists a scalar invariant φ such that $\varphi(\exp(sX'))$ is unbounded as $s \to s^*$. Now, for all $s < s^*$, we can define $h(\exp(sX)) := \exp(sX')$, and this gives an extension h of f_{s_1} which is analytic (since it commutes with the exponential maps, which are analytic). By Lemma 1.4.6, h is in fact an isometric immersion. By definition of scalar invariants we have

$$\varphi(\exp(sX')) = \tilde{\varphi}(\exp(sX))\,,$$

where $\tilde{\varphi}$ is the invariant in $(\mathscr{M}, g)$ corresponding to φ. But this is not possible since $\tilde{\varphi}(\exp(sX))$ has a finite limit when $s \to s^*$, and provides the desired contradiction. $\square$

From the last lemma we deduce that there exists a unique element, say $h(y)$, in a normal neighborhood of $f_{s_1}(c(s_1))$ in $\mathscr{M}'$ such that $h(y) = \exp(X')$. Hence, we have extended f_{s_1} to a map h defined on $V \cap I^{\pm}(c(s_1))$. In fact, h is also an isometric immersion, by the same argument as above, since it commutes with the exponential maps of $\mathscr{M}$ and $\mathscr{M}'$. Then, since the curve c is timelike, this is sufficient to conclude that we can do the analytic continuation beyond $c(\bar{s})$, since $V \cap I^{\pm}(c(s_1))$ is an open set, and thus contains a segment of the geodesic $c(s)$, for s in a neighborhood of $\bar{s}$.

Let us consider now a general, not necessarily timelike, continuous curve $c(s), 0 \le s \le 1$, with $c(0) \in U$. As before, we consider the set:

$$\{s \in [0,1] \mid \text{there exists an analytic continuation of } f_U \text{ along } c(s'), 0 \le s' \le s\}, \tag{1.4.6}$$

and its supremum $\tilde{s}$. Assume that $\tilde{s}$ is not reached. Let again W be a normal neighborhood of $c(\tilde{s})$ such that every point of W contains a normal neighborhood which contains W. Then, let z be an element of the set $I^+(c(\tilde{s})) \cap W$. $I^-(z) \cap W$ is therefore an open set in W containing $c(\tilde{s})$. Hence we can choose $s_1 < \tilde{s}$ such that the curve segment $c([s_1, \tilde{s}])$ is included in $I^-(z) \cap W$, see Figure 1.4.1. In particular, $z \in I^+(c(s_1)) \cap W$. Since there exists an analytic

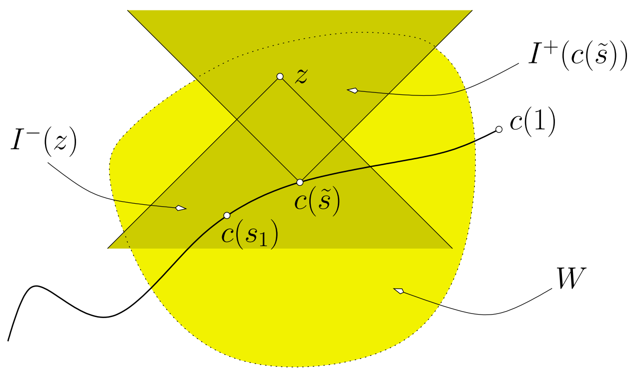

Figure 1.4.1: The analytic continuation at $c(\tilde{s})$.

continuation up to $c(s_1)$, we have an isometric immersion f_{s_1} defined on a neighborhood U_{s_1} of $c(s_1)$, which can be assumed to be included in W. Hence, from what has been seen previously, f_{s_1} can be extended as an isometric immersion, ψ_1, on $U_z := U_{s_1} \cup (I^+(c(s_1)) \cap W)$, which contains z. We now do the same operation for ψ_1 on U_z: we can extend it by analytic continuation to an isometric immersion ψ_2 defined on $U_z \cup (I^-(z) \cap W)$, which is an open set containing the entire segment of the curve x between $c(s_1)$ and $c(\tilde{s})$. In particular, ψ_1 and ψ_2 coincide on U_z, i.e. on their common domain of definition; thus we obtain an analytic continuation of f_{s_1} along the curve $c(s)$, for $s_1 \leq s \leq \tilde{s}$; this continuation also coincides with the continuation $f_s,\ s \in [s_1, \tilde{s}[$. This is in contradiction with the assumption that $\tilde{s}$ is not reached by any analytic continuation from f_U along x. Hence $\tilde{s} = 1$ and is reached, that is to say we have proved the existence of an analytic continuation of f_U along all the curve x.

The remaining arguments are as in [172]. □

1.5 The Reissner-Nordström metrics

The Reissner-Nordström metrics are the unique spherically symmetric solutions of the Einstein-Maxwell equations with vanishing cosmological constant. They turn out to be static, asymptotically flat, and describe black hole spacetimes with interesting global properties for a certain range of parameters. The metric takes the form

$$ {}^4g = -\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2\,, \tag{1.5.1} $$

where m is, as usual, the ADM mass of g and Q is the total electric charge. The electromagnetic potential takes the form

$$ A = \frac{Q}{r}dt\,. \tag{1.5.2} $$

The equation $g(\partial_t, \partial_t) = 0$ has solutions $r = r_\pm$ provided that $|Q| \leq m$:

$$ r_\pm = m \pm \sqrt{m^2 - Q^2}\,. $$

These hypersurfaces become Killing horizons, or bifurcate Killing horizons, in suitable extensions of the Reissner-Nordström metric.

Calculating as in Example 1.3.9, p. 41, one finds that the surface gravities of the Killing horizons $r = r_\pm$ of the Reissner-Nordström metric equal

$$ \begin{aligned} \kappa_\pm &= -\frac{1}{2}\partial_r g_{tt}|_{r=r_\pm} = \frac{1}{2}\partial_r\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\Big)\Big|_{r=r_\pm} = \frac{mr_\pm - Q^2}{r_\pm^3} \\ &= \pm\frac{\sqrt{m^2 - Q^2}}{r_\pm^2}\,. \end{aligned} $$

For $r = r_+$ this is strictly positive unless $|Q| = m$; so we see that Reissner-Nordström black holes are non-degenerate for $|Q| < m$, and degenerate when $|Q| = m$.

The global structure of a class of maximal extensions of non-degenerate Reissner-Nordström spacetimes is presented in Example 4.3.1, p. 141, while that of degenerate solutions can be found in Example 4.3.6, p. 147.

INCIDENTALLY: Suppose that the metric (1.5.1) models an electron, for which

$$m_e \approx 9.11 \times 10^{-31}\text{kg}\,, \quad Q_e \approx -1.60 \times 10^{-19}\text{C}\,.$$

Our form of the metric requires units in which $G/c^2 = 1$ and $G/(4\pi\epsilon_0 c^4) = 1$. Using

$$\frac{1}{4\pi\epsilon_0} \equiv k_e \approx 8.99 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}\,, \quad G \approx 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$$

we find

$$m_e \approx 6.75 \times 10^{-58}\text{m} \times \frac{c^2}{G}\,, \quad Q_e \approx -1.38 \times 10^{-36}\text{m} \times \sqrt{\frac{4\pi\epsilon_0 c^4}{G}}\,,$$

leading to

$$\frac{|Q_e|}{m_e} \approx 2.04 \times 10^{21}\,.$$

We see that a point electron is then described by a naked singularity.

For a proton we have instead

$$m_p \approx 1.67 \times 10^{-27}\text{kg}\,,$$

with the charge $Q_p = -Q_e$, which gives

$$\frac{Q_p}{m_p} \approx 1.11 \times 10^{18}\,.$$

The difference is, however, that the proton is not a point particle, so the Reissner-Nordström metric applies, at best, only outside the charge radius of the proton $r_p \approx 0.85$ fm. □

In dimensions $n+1 \geq 5$ one has [214] the following counterpart of (1.5.1)-(1.5.2):

$$^{n+1}g = -\Big(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\Big)dt^2 + \frac{dr^2}{1 - \dfrac{2m}{r^{n-2}} + \dfrac{Q^2}{r^{2(n-2)}}} + r^2 d\Omega^2\,, \tag{1.5.3}$$

$$A = \frac{Q}{r^{n-2}}dr\,, \tag{1.5.4}$$

where m is related to the ADM mass, and Q to the total charge.

INCIDENTALLY: The RN metrics have the interesting property of being *timelike geodesically complete*, but not *null geodesically complete*. To see that, consider a timelike geodesic γ parameterised by proper time, thus we have

$$-1 = -\Big(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\Big)\dot{t}^2 + \frac{\dot{r}^2}{1 - \dfrac{2m}{r^{n-2}} + \dfrac{Q^2}{r^{2(n-2)}}} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)\,.$$

Conservation of "energy", $g(\dot{\gamma}, \partial_t) = -E$, implies that

$$\Big(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\Big)\dot{t} = E\,,$$

hence

$$-1 = \frac{\dot{r}^2 - E^2}{1 - \dfrac{2m}{r^{n-2}} + \dfrac{Q^2}{r^{2(n-2)}}} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)\,. \tag{1.5.5}$$

Equivalently,

$$\begin{aligned}\dot r^2 &= E^2 - \left(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\right) \underbrace{\left(1 + r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2)\right)}_{\geq 1} \\ &\leq E^2 - \left(1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}\right). \qquad (1.5.6)\end{aligned}$$

As r approaches zero the right-hand side becomes negative, which is not possible. It follows that timelike geodesics cannot approach $r = 0$. It is then not too difficult to prove that maximally extended timelike geodesics are complete in the extensions of Figures 4.3.1, p. 142 and 4.3.6, p. 148, and timelike geodesic completeness follows.

Obvious modifications of the above calculation similarly show that null geodesics with non-zero angular momentum, $g(\dot\gamma, \partial_\varphi) \neq 0$, cannot reach the singular boundary $\{r = 0\}$ and are complete.

On the other hand, radial null geodesics reach $r = 0$ in finite affine parameter: For then we have zero at the left-hand side of (1.5.5), without an angular-momentum contribution, giving

$$\dot r = \pm E \quad \Longrightarrow \quad r(s) - r_0 = \pm E(s - s_0)\,.$$

Hence null radial geodesics reach $r = 0$ in finite affine time either to the future or to the past, showing null geodesic incompleteness. □

1.6 The Kerr metric

The Kerr family of metrics provide a “rotating generalisation” of the Schwarzschild metric. Its importance stems from the *black hole uniqueness theorems*, which establish uniqueness of Kerr black holes under suitable global conditions (cf., e.g., [72] and references therein). It should, however, be kept in mind that the Schwarzschild metric describes not only spherically symmetric black holes, but also the *vacuum exterior region of any spherically symmetric* matter configuration. There is no such universality property for stationary axi-symmetric configurations. Indeed, the construction of axisymmetric stationary stellar models is a rather complicated undertaking, we refer the reader to [202] for more information about the subject.

As such, the two parameter family of *Kerr metrics* in Boyer-Lindquist coordinates take the form

$$\begin{aligned}g = -dt^2 &+ \frac{2mr}{\Sigma}(dt - a\sin^2(\theta)d\varphi)^2 \\ &+ (r^2 + a^2)\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,. \qquad (1.6.1)\end{aligned}$$

Here

$$\Sigma = r^2 + a^2\cos^2(\theta)\,, \qquad \Delta = r^2 + a^2 - 2mr = (r - r_+)(r - r_-)\,, \quad (1.6.2)$$

and $r_+ < r < \infty$, where

$$r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}}\,.$$

The metric satisfies the vacuum Einstein equations for any values of the parameters a and m, but we will mainly consider parameters in the range

$$0 < |a| \leq m\,.$$

The case $|a| = m$ will sometimes require separate consideration, as then Δ acquires one double root at $r = m$, instead of two simple ones. When $a = 0$, the Kerr metric reduces to the Schwarzschild metric, and therefore does not need to be discussed any further. The case $a < 0$ can be reduced to $a > 0$ by changing φ to $-\varphi$; this corresponds to a change of the direction of rotation. There is therefore no loss of generality to assume that $a > 0$, which will be done whenever the sign of a matters for the discussion at hand. The Kerr metrics with $|a| > m$ can be shown to be "nakedly singular" (compare (1.6.5) below), whence our lack of interest in those solutions.

It is straightforward to check that the metric (1.6.1) reduces to the Schwarzschild one when $a = 0$. It turns out that the case $m = 0$ leads to Minkowski spacetime: For $a = 0$ this is obvious; for $a \neq 0$ the coordinate transformation (cf., e.g., [45, p. 102])

$$R^2 = r^2 + a^2\sin^2(\theta)\,, \qquad R\cos(\Theta) = r\cos(\theta)\,, \tag{1.6.3}$$

brings g to the Minkowski metric η in spherical coordinates:

$$\eta = -dt^2 + dR^2 + R^2\left(d\Theta^2 + \sin^2(\Theta)d\varphi^2\right)\,. \tag{1.6.4}$$

As $m = 0$ turns out to be Minkowski, and $a = 0$ Schwarzschild, it is customary to interpret m as a parameter related to mass, and a as a parameter related to rotation. This can be made precise by calculating the total mass and angular momentum of the solution using e.g. Hamiltonian methods. One then finds that m is indeed the total mass, while

$$J = ma$$

is the component of the total angular momentum in the direction of the axis of rotation $\sin(\theta) = 0$.

Incidentally: It might be of interest to put some numbers in. Consider, for instance the sun. As such, there are several ways of calculating the total angular momentum $J_\odot$ of our nearest stellar neighbour, see [158] for a discussion of the various estimates and their discrepancies. If we choose the averaged value [158]

$$J_\odot \approx 1.92 \times 10^{41}\ \text{kg m}^2\ \text{s}^{-1}$$

for the angular momentum of the sun, and keep in mind the estimate $M_\odot \approx 1.99 \times 10^{30}$kg for the mass of the sun (cf., e.g., `http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html`), we find

$$a_\odot = \frac{J_\odot}{M_\odot} \approx 0.96 \times 10^{10}\text{m}^2\text{s}^{-1} \approx 322\,\text{m} \times c\,,$$
$$\frac{M_\odot G}{c^2} \approx 1.48\,\text{km}\,, \qquad \frac{a_\odot}{M_\odot} \approx 0.22\frac{G}{c}\,.$$

Keeping in mind that the units used in (1.6.1) are such that $G = c = 1$, we see that $|a| < m$ for the sun, with both values of a and m being of the same order.

If we consider the earth to be a rigidly rotating uniform sphere, the corresponding numbers are

$$J_{\oplus} \approx 7.10 \times 10^{33} \text{ kg m}^2 \text{ s}^{-1}, \quad M_{\oplus} \approx 5.98 \times 10^{24}\text{kg}, \quad a_{\oplus} \approx 3.96\,\text{m} \times c,$$
$$\frac{M_{\oplus} G}{c^2} \approx 0.44\,\text{cm}, \qquad \frac{a_{\oplus} c}{M_{\oplus} G} \approx 890\,.$$

We conclude that if the earth collapsed to a Kerr metric without shedding angular-momentum, a naked singularity would result. □

The metric (1.6.1) is not defined at points where Σ vanishes:

$$\Sigma = 0 \quad \Longleftrightarrow \quad r = 0\,, \ \cos\theta = 0\,.$$

There is a "real singularity on Σ", in the sense that the metric cannot be extended across this set in a C^2 manner. The standard argument for this in the literature invokes the Kretschmann scalar (cf., e.g., [181])

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2(r^2 - a^2\cos^2\theta)(\Sigma^2 - 16a^2r^2\cos^2(\theta))}{\Sigma^6}\,, \tag{1.6.5}$$

which is unbounded when the set $\{\Sigma = 0\}$ is approached from most directions.

Now, this does not quite settle the issue because $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 0$ e.g. on all curves approaching $\{\Sigma = 0\}$ with $r^2 = a^2\cos^2\theta$. Similarly, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ remains bounded on curves on which either $r^2 - a^2\cos^2(\theta)$ or $\Sigma^2 - 16a^2r^2\cos^2(\theta)$ go to zero sufficiently fast. So, one can imagine that spacetime could nevertheless be extended along some clever family of curves approaching Σ in a specific way.

This problem is unfortunately not cured by considering the length of the Killing vector ∂_t,

$$g(\partial_t, \partial_t) = -1 + \frac{2mr}{\Sigma}\,,$$

which again tends to infinity as Σ is approached from most directions. This only implies inextendibility "along most directions" at $\{\Sigma = 0\}$ by Theorem 1.4.2, p. 57.

It turns out that the issue can be resolved by a result of Carter [43, p. 1570] (compare [225, Proposition 4.5.1]), which we quote here without proof:

Proposition 1.6.2 *Causal geodesics accumulating at* $\{\Sigma = 0\}$ *lie entirely in the equatorial plane* $\{\cos\theta = 0\}$. □

Now, on the equatorial plane we have

$$g(\partial_t, \partial_t)|_{\cos\theta=0} = -1 + \frac{2m}{r}\,,$$

which is unbounded on any curve approaching $\{\Sigma = 0\}$. We can therefore invoke Proposition 1.4.3, p. 57, to conclude that, indeed, no extensions are possible through $\{\Sigma = 0\}$.

Equation (1.6.3) suggests that the topology of the singular set has something to do with a ring, though it is not clear how to make a precise statement to this effect.

The Kerr metric is stationary with the Killing vector field $X = \partial_t$ generating asymptotic time translations, as well as axisymmetric with the Killing vector field $Y = \partial_\varphi$ generating rotations.

The metric components $g_{\mu\nu}$ can be read-off from the expanded version of (1.6.1):

$$\begin{aligned} g = & -\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 - \frac{4amr\sin^2\theta}{\Sigma}dtd\varphi + \\ & + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \,. \end{aligned} \tag{1.6.6}$$

REMARK 1.6.3 An elegant way of associating global invariants to Killing vectors X is provided by Komar integrals, which are integrals of the form

$$\int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} \,, \tag{1.6.7}$$

where R and T are constants, and $dS_{\alpha\beta}$ form a basis of the space of two-forms defined as

$$dS_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}dx^\gamma \wedge dx^\delta \,. \tag{1.6.8}$$

A key property of (1.6.7) is that in vacuum, and with zero cosmological constant, the integrals are independent of r and t. This follows from the divergence theorem together with the identity (compare (A.21.8), p. 308)

$$\nabla_\alpha \nabla^\alpha X^\beta = R^\alpha{}_\beta X^\beta \,.$$

In Kerr spacetime it is of interest to calculate (1.6.7) for both Killing vectors $X = \partial_t$ and $X = \partial_\varphi$. In order to do the calculation for both vectors at once let us denote either ∂_t or ∂_φ by ∂_λ, hence $X_\mu = g_{\mu\lambda}$. For the calculations we need the inverse metric, the components of which are

$$\begin{aligned} g^{tt} &= -1 - \frac{4mr\left(a^2+r^2\right)}{\left(a^2+r(r-2m)\right)\left(\cos(2\theta)a^2+a^2+2r^2\right)} \,, \\ g^{rr} &= \frac{a^2-2mr+r^2}{a^2\cos^2(\theta)+r^2} \,, \qquad g^{\theta\theta} = \frac{1}{a^2\cos^2(\theta)+r^2} \,, \\ g^{\varphi\varphi} &= \frac{\csc^2(\theta)\left(a^2\cos(2\theta)+a^2+2r(r-2m)\right)}{\left(a^2+r(r-2m)\right)\left(a^2\cos(2\theta)+a^2+2r^2\right)} \,, \\ g^{t\varphi} &= -\frac{4amr}{\left(a^2+r(r-2m)\right)\left(a^2\cos(2\theta)+a^2+2r^2\right)} \,. \end{aligned} \tag{1.6.9}$$

Then

$$\begin{aligned} \int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} &= \int_{r=R,t=T} \nabla^{[\alpha} X^{\beta]} dS_{\alpha\beta} = \int_{r=R,t=T} \nabla_{[\nu} X_{\mu]} g^{\mu\alpha} g^{\nu\beta} dS_{\alpha\beta} \\ &= \int_{r=R,t=T} \partial_{[\nu} X_{\mu]} g^{\mu\alpha} g^{\nu\beta} dS_{\alpha\beta} = 2\int_{r=R,t=T} \partial_{[\nu} X_{\mu]} g^{\mu t} g^{\nu r} dS_{tr} \\ &= 2\int_{r=R,t=T} \partial_{[\nu} g_{\mu]\lambda} g^{\mu t} g^{\nu r} dS_{tr} = \int_{r=R,t=T} (\partial_r g_{\mu\lambda} - \partial_\mu g_{r\lambda}) g^{\mu t} g^{rr} dS_{tr} \\ &= \int_{r=R,t=T} \partial_r g_{\mu\lambda} g^{\mu t} g^{rr} dS_{tr} = \int_{r=R,t=T} (\partial_r g_{t\lambda} g^{tt} + \partial_r g_{\varphi\lambda} g^{\varphi t}) g^{rr} dS_{tr} \,. \end{aligned}$$

As a by-product of T- and R-independence of (1.6.7), one can calculate the integrals by passing to the limit $R \to \infty$, which simplifies the calculations considerably. Thus

$$\int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} = \lim_{R\to\infty} \int_{r=R,t=T} (\partial_r g_{t\lambda} g^{tt} + \partial_r g_{\varphi\lambda} g^{\varphi t}) g^{rr} r^2 \underbrace{\sin(\theta) d\theta\, d\varphi}_{=:d^2\mu} . \tag{1.6.10}$$

To finish the calculation we need the asymptotic behaviour of the metric functions for large r. We find:

$$\begin{aligned} g = &- \left(1 - \frac{2m}{r} + O(r^{-2})\right) dt^2 - \left(\frac{4am}{r} + O(r^{-2})\right) \sin^2\theta\, dt\, d\varphi \\ &+ \left(r^2 + O(1)\right) \sin^2\theta d\varphi^2 + \left(1 + \frac{2m}{r} + O(r^{-2})\right) dr^2 \\ &+ \left(r^2 + O(1)\right) d\theta^2 . \end{aligned} \tag{1.6.11}$$

This shows explicitly asymptotic flatness of the metric. For the inverse metric, one obtains

$$\begin{aligned} g^{tt} = -1 - \frac{2m}{r} + O(r^{-2}) , \quad g^{rr} = 1 - \frac{2m}{r} + O(r^{-2}) , \quad g^{\theta\theta} = \frac{1}{r^2} + O(r^{-3}) , \\ g^{\varphi\varphi} = \frac{1}{\sin^2(\theta) r^2} + O(r^{-3}) , \quad g^{t\varphi} = -\frac{2am}{r^3} + O(r^{-4}) . \end{aligned} \tag{1.6.12}$$

We are ready to return to (1.6.10):

$$\int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} = \lim_{R\to\infty} \int_{r=R,t=T} (\partial_r g_{t\lambda} g^{tt} + \partial_r g_{\varphi\lambda} g^{\varphi t}) r^2 d^2\mu . \tag{1.6.13}$$

When $X = \partial_t$ this becomes

$$\begin{aligned} \int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} &= \lim_{R\to\infty} \int_{r=R,t=T} (-\partial_r g_{tt} + \partial_r g_{\varphi t} g^{\varphi t}) r^2 d^2\mu \\ &= -\lim_{R\to\infty} \int_{r=R,t=T} \partial_r g_{tt} r^2 d^2\mu = 8\pi m . \end{aligned} \tag{1.6.14}$$

When $X = \partial_\varphi$ we obtain instead

$$\begin{aligned} \int_{r=R,t=T} \nabla^\alpha X^\beta dS_{\alpha\beta} &= \lim_{R\to\infty} \int_{r=R,t=T} (-\partial_r g_{t\varphi} + \partial_r g_{\varphi\varphi} g^{\varphi t}) r^2 d^2\mu \\ &= \lim_{R\to\infty} \int_{r=R,t=T} (-\partial_r g_{t\varphi} + \partial_r g_{\varphi\varphi} g^{\varphi t}) r^2 d^2\mu \\ &= -12\pi am \int_0^\pi \sin^3(\theta)\, d\theta = -16\pi am . \end{aligned} \tag{1.6.15}$$

□

Because of the occurrence of the function Δ in the denominator of g_{rr}, the metric (1.6.6) is singular at $r = r_\pm$. Similarly to the Schwarzschild case, it turns out that the metric can be smoothly extended both across $r = r_+$ and $r = r_-$, with the sets

$$\mathscr{H}_\pm := \{r = r_\pm\}$$

being smooth null hypersurfaces in the extension.

INCIDENTALLY: Higher dimensional generalisations of the Kerr metric have been constructed by Myers and Perry [214].

We will give an extended discussion of a family of maximal analytic extensions of the Kerr metric, and their global structure, in Section 4.7.3, p. 167. As a first step towards this we consider the extension obtained by replacing t with a new coordinate

$$v = t + \int \frac{r^2 + a^2}{\Delta} dr \,, \tag{1.6.16}$$

with a further replacement of φ by

$$\phi = \varphi + \int \frac{a}{\Delta} dr \,. \tag{1.6.17}$$

It is convenient to use the symbol $\hat{g}$ for the metric g in the new coordinate system, obtaining

$$\begin{aligned} \hat{g} \;\; = \;\; & -\Big(1 - \frac{2mr}{\Sigma}\Big) dv^2 + 2dr dv + \Sigma d\theta^2 - 2a \sin^2(\theta) d\phi dr \\ & + \frac{(r^2 + a^2)^2 - a^2 \sin^2(\theta) \Delta}{\Sigma} \sin^2(\theta) d\phi^2 - \frac{4amr \sin^2(\theta)}{\Sigma} d\phi dv \end{aligned} \tag{1.6.18}$$

In order to see that (1.6.18) provides a smooth Lorentzian metric for $v \in \mathbb{R}$ and $r \in (0, \infty)$, note first that the coordinate transformation (1.6.16)-(1.6.17) has been tailored to remove the $1/\Delta$ singularity in (1.6.6), so that all coefficients are now analytic functions on $\mathbb{R} \times (0, \infty) \times S^2$. A direct calculation of the determinant of $\hat{g}$ is somewhat painful, a simpler way is to proceed as follows: first, the calculation of the determinant of the metric (1.6.6) reduces to that of a two-by-two determinant in the (t, ψ) variables, leading to

$$\det g = -\sin^2(\theta) \Sigma^2 \,. \tag{1.6.19}$$

Next, it is very easy to check that the determinant of the Jacobi matrix

$$\partial(v, r, \theta, \phi) / \partial(t, r, \theta, \varphi)$$

equals one. It follows that $\det \hat{g} = -\sin^2(\theta) \Sigma^2$ for $r > r_+$. Analyticity implies that this equation holds globally, which (since Σ has no zeros) establishes the Lorentzian signature of $\hat{g}$ for all positive r.

Let us show that the region $r < r_+$ is a black hole region, in the sense of (1.2.12). We start by noting that ∇r is a causal vector for $r_- \leq r \leq r_+$. A direct calculation using (1.6.18) is again somewhat lengthy, instead we use (1.6.6) in the region $r > r_+$ to obtain there

$$\hat{g}(\nabla r, \nabla r) = g(\nabla r, \nabla r) = g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\Sigma} = \frac{(r - r_+)(r - r_-)}{r^2 + a^2 \cos^2 \theta} \,. \tag{1.6.20}$$

But the left-hand side of this equation is an analytic function throughout the extended manifold $\mathbb{R} \times (0, \infty) \times S^2$, and uniqueness of analytic extensions implies that $\hat{g}(\nabla r, \nabla r)$ equals the expression at the extreme right of (1.6.20) throughout.

(The intermediate equalities have only been assumed to be valid for $r > r_+$ in the calculation above, since g has only been defined for $r > r_+$.) Thus ∇r is spacelike if $r < r_-$ or $r > r_+$, null on the hypersurfaces $\{r = r_\pm\}$ (called *"Killing horizons"*, see Section 1.3.2), and timelike in the region $\{r_- < r < r_+\}$; note that this last region is empty when $|a| = m$.

We choose a time orientation so that ∇t is past pointing in the region $r > r_+$. Keeping in mind our signature of the metric, this means that t increases on future directed causal curves in the region $r > r_+$.

Suppose, now, that $a^2 < m^2$, and consider a future directed timelike curve $\gamma(s)$ that starts in the region $r > r_+$ and enters the region $r < r_+$. Since $\dot\gamma$ is timelike it meets the null hypersurface $\{r = r_+\}$ transversally, and thus r is decreasing along γ at least near the intersection point. As long as γ stays in the region $\{r_- < r < r_+\}$ the scalar product $g(\dot\gamma, \nabla r)$ has constant sign, since both $\dot\gamma$ and ∇r are timelike there. But

$$\frac{dr}{ds} = \dot\gamma^i \nabla_i r = g_{ij}\dot\gamma^i \nabla^j r = g(\dot\gamma, \nabla r)\,, \tag{1.6.21}$$

and dr/ds is negative near the entrance point. We conclude that dr/ds is negative along such γ's on $\{r_- < r < r_+\}$. This implies that r is strictly decreasing along future directed causal curves in the region $\{r_- < r < r_+\}$, so that such curves can only leave this region through the set $\{r = r_-\}$. In other words, no causal communication is possible from the region $\{r < r_+\}$ to the "exterior world" $\{r > r_+\}$ in the extension that we constructed so far.

The Schwarzschild metric has the property that the set $g(X, X) = 0$, where X is the "static Killing vector" ∂_t, coincides with the event horizon $r = 2m$. This is not the case any more for the Kerr metric, where we have

$$g(\partial_t, \partial_t) = \hat g(\partial_v, \partial_v) = \hat g_{vv} = -\Big(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\Big)\,.$$

The equation $\hat g(\partial_v, \partial_v) = 0$ defines instead a set called the *ergosphere*:

$$\hat g(\partial_v, \partial_v) = 0 \quad \Longleftrightarrow \quad \mathring{r}_\pm = m \pm \sqrt{m^2 - a^2\cos^2\theta}\,,$$

see Figures 1.6.1 and 1.6.2. The ergosphere touches the horizons at the axes of symmetry $\cos\theta = \pm 1$. Note that $\partial\mathring{r}_\pm/\partial\theta \neq 0$ at those axes, so the ergosphere has a cusp there. The region bounded by the outermost horizon $r = r_+$ and the outermost ergosphere $r = \mathring{r}_+$ is called the *ergoregion*, with X spacelike in its interior.

It is important to realise that the ergospheres

$$\mathscr{E}_\pm := \{r = \mathring{r}_\pm\}$$

are *not* Killing horizons for the Killing vector ∂_t. Recall that part of the definition of a Killing horizon $\mathscr{H}$ is the requirement that $\mathscr{H}$ is a *null* hypersurface. But this is not the case for $\mathscr{E}_\pm$: Indeed, note that the Killing vectors ∂_φ and ∂_t are both tangent to $\mathscr{E}_\pm$, and thus are all their linear combinations. Now, the

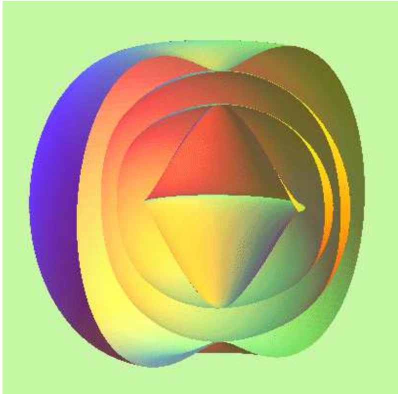

Figure 1.6.1: A coordinate representation [233] of the outer ergosphere $r = \mathring{r}_+$, the event horizon $r = r_+$, the Cauchy horizon $r = r_-$, and the inner ergosphere $r = \mathring{r}_-$ with the singular ring in Kerr spacetime. Computer graphics by Kayll Lake [181].

character of the principal orbits of the isometry group $\mathbb{R} \times U(1)$ is determined by the sign of the determinant

$$\det \begin{pmatrix} g_{tt} & g_{t\varphi} \\ g_{t\varphi} & g_{\varphi\varphi} \end{pmatrix} = -\Delta \sin^2(\theta)\,. \tag{1.6.22}$$

Therefore, when $\sin\theta = 0$ the orbits are either null or one-dimensional, while for $\theta \neq 0$ the orbits are timelike in the regions where $\Delta > 0$, spacelike where $\Delta < 0$ and null where $\Delta_r = 0$. Thus, at every point of $\mathscr{E}_\pm$ except at the intersection with the axis of rotation there exist linear combinations of ∂_t and ∂_φ which are timelike. This implies that these hypersurfaces are *not* null, as claimed.

We refer the reader to Refs. [43] and [225] for an exhaustive analysis of the geometry of the Kerr spacetime.

INCIDENTALLY: One of the most useful methods for analysing solutions of wave equations is the *energy method*. As an illustration, consider the wave equation

$$\Box u = 0\,. \tag{1.6.23}$$

Let $\mathscr{S}_t$ is a foliation of $\mathscr{M}$ by spacelike hypersurfaces, the energy E_t of u on $\mathscr{S}_t$ associated to a vector field X is defined as

$$E(t) = \int_{\mathscr{S}_t} T^\mu{}_\nu X^\mu \eta_\nu\,,$$

where $T_{\mu\nu}$ is the usual energy-momentum tensor of a scalar field,

$$T_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2}\nabla^\alpha u \nabla_\alpha u\, g_{\mu\nu}\,.$$

The energy functional E has two important properties: 1): $E \geq 0$ if X is causal, and 2): $E(t)$ is conserved if X is a Killing vector field and, say, u has compact support on each of the $\mathscr{S}_t$.

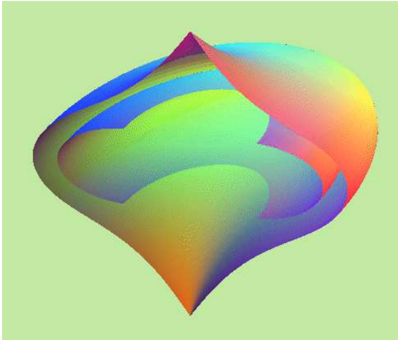

Figure 1.6.2: Isometric embedding in Euclidean three space of the ergosphere (the outer hull), and part of the event horizon, for a rapidly rotating Kerr solution. The hole in the event horizon arises because there is no global isometric embedding for the event horizon when $a/m > \sqrt{3}/2$ [233]. Somewhat surprisingly, the embedding fails to represent accurately the fact that the cusps at the rotation axis are pointing inwards, and not outwards. Computer graphics by Kayll Lake [181].

Now, the existence of ergoregions where the Killing vector X becomes spacelike leads to an $E(t)$ which is not necessarily positive any more, and the energy stops being a useful tool in controlling the behavior of the field. This is one of the obstacles to our understanding of both linear and non-linear, solutions of wave equations on a Kerr background[12], not to mention the wide open question of non-linear stability of the Kerr black holes within the class of globally hyperbolic solutions of the vacuum Einstein equations. □

The hypersurfaces

$$\mathscr{H}_\pm := \{r = r_\pm\}$$

provide examples of *null acausal boundaries*. Because $g(\nabla r, \nabla r)$ vanishes at $\mathscr{H}_\pm$, the usual calculation (see Proposition A.13.2, p. 271) shows that the integral curves of ∇r with $r = r_\pm$ are null geodesics. Such geodesics, tangent to a null hypersurface, are called *generators* of this hypersurface. A direct calculation of ∇r from (1.6.18) requires work which can be avoided as follows: in the coordinate system (t, r, θ, φ) of (1.6.6) one obtains immediately

$$\nabla r = g^{\mu\nu}\partial_\mu r \partial_\nu = \frac{\Delta}{\Sigma}\partial_r \,.$$

Now, under (1.6.16)-(1.6.17) the vector ∂_r transforms as

$$\partial_r \to \partial_r + \frac{a}{\Delta}\partial_\phi + \frac{r^2 + a^2}{\Delta}\partial_v \,.$$

More precisely, if we use the symbol $\hat{r}$ for the coordinate r in the coordinate system (v, r, θ, ϕ), and retain the symbol r for the coordinate r in the coordinates

[12]See [25, 101] and refs. therein for further information on that subject.

(t, r, θ, φ), we have

$$\partial_r = \frac{\partial \hat{r}}{\partial r}\partial_{\hat{r}} + \frac{\partial \phi}{\partial r}\partial_\phi + \frac{\partial v}{\partial r}\partial_v = \partial_{\hat{r}} + \frac{a}{\Delta}\partial_\phi + \frac{r^2+a^2}{\Delta}\partial_v \,.$$

Forgetting the hat over r, we see that in the coordinates (v, r, θ, ϕ) we have

$$\nabla r = \frac{1}{\Sigma}(\Delta\partial_r + a\partial_\phi + (r^2+a^2)\partial_v) \,.$$

Since Δ vanishes at $r = r_\pm$, and $r^2 + a^2$ equals $2mr_\pm$ there, we conclude that the "stationary-rotating" Killing field $X + \Omega_+ Y$, where

$$X := \partial_t \equiv \partial_v \,, \quad Y := \partial_\phi \equiv \partial_\varphi \,, \quad \Omega_+ := \frac{a}{2mr_+} \equiv \frac{a}{a^2 + r_+^2} \,, \tag{1.6.24}$$

is proportional to ∇r on $\{r > r_+\}$:

$$X + \Omega_+ Y = \partial_v + \frac{a}{2mr_+}\partial_\phi = \frac{\Sigma}{a^2 + r_+^2}\nabla r \quad \text{on} \quad \mathscr{H}_+ \,.$$

It follows that $\partial_t + \Omega_+\partial_\varphi$ is null and *tangent* to the generators of the horizon $\mathscr{H}_+$. In other words, the generators of $\mathscr{H}_+$ are rotating with respect to the frame defined by the stationary Killing vector field X. This property is at the origin of the definition of Ω_+ as the *angular velocity* of the event horizon.

1.6.1 Non-degenerate solutions ($a^2 < m^2$): Bifurcate horizons

The study of the global structure of Kerr is somewhat more involved than those already encountered. An obvious second extension of the coordinate system of (1.6.6) is obtained when t is replaced by a new coordinate[13]

$$u = t - \int_{r_+}^{r} \frac{r^2+a^2}{\Delta} dr \,, \tag{1.6.25}$$

with a further replacement of φ by

$$\psi = \varphi - \int_{r_+}^{r} \frac{a}{\Delta} dr \,. \tag{1.6.26}$$

If we use the symbol $\tilde{g}$ for the metric g in the new coordinate system, we obtain

$$\begin{aligned} \tilde{g} \quad &= \quad -\Big(1 - \frac{2mr}{\Sigma}\Big)du^2 - 2drdu + \Sigma d\theta^2 + 2a\sin^2(\theta)d\psi dr \\ &+ \frac{(r^2+a^2)^2 - a^2\sin^2(\theta)\Delta}{\Sigma}\sin^2(\theta)d\psi^2 - \frac{4amr\sin^2(\theta)}{\Sigma}d\psi du \,. \end{aligned} \tag{1.6.27}$$

In Schwarzschild one replaces (t, r) by (u, v), and with a little further work a well behaved extension is obtained. It should be clear that this shouldn't be that simple for the Kerr metric, because the two extensions constructed so far

[13] The discussion here is based on [32, 43]. I am grateful to Julien Cortier for useful discussions concerning this section.

involve incompatible redefinitions of the angular variable φ, compare (1.6.17) and (1.6.26).

The calculations that follow are essentially a special case of the general construction of Rácz and Wald [239], presented in Theorem 1.7.3, p. 89 below, where the argument is traced back to the fact that the surface gravity is constant on the horizons $\{r = r_\pm\}$.

Now, recall that we have seen in Remark 1.2.12, p. 26, how to regularise two-dimensional Lorentzian metrics with a singularity structure as in (1.2.43). The first step of the calculation there gets rid of the zero in the denominator of g_{rr} *provided that there is a first order zero in* g_{tt} *at that point*; then it is easy to remove the multiplicative first order zero in g_{uv} by a logarithmic transformation to the variables $\hat{u}$ and $\hat{v}$ as in (1.2.45). Note that a first-order zero requires

$$a^2 < m^2 \,,$$

which we are going to assume in the remainder of this section; this is not an *ad-hoc* restriction, as the geometry of the spacetime is essentially different in the *extreme case* $a^2 = m^2$, see the last sentence of Section 1.6.6.

So, under the current conditions the Kerr metric has a first order pole in g_{rr} at $r = r_\pm$, but there is no zero in g_{tt} at those values of r. The trick is to change φ, near $r = r_\pm$, to a new angular variable

$$\varphi_\pm = \varphi - \frac{a}{2m\alpha_\pm} t \,, \tag{1.6.28}$$

choosing the free constants $\alpha_\pm \neq 0$ so that the new g_{tt} vanishes at $r_\pm$. Indeed, after tedious but otherwise straightforward algebra, in the coordinate system $(t, r, \theta, \varphi_\pm)$ the metric (1.6.6) takes the form

$$g = \frac{\Sigma}{\Delta} dr^2 + g_{tt} dt^2 + 2 g_{\varphi_\pm t} d\varphi_\pm dt + \left(r^2 + a^2 + \frac{2mar}{\Sigma} \right) \sin^2(\theta) d\varphi_\pm^2 + \Sigma d\theta^2 \,, \tag{1.6.29}$$

with

$$\begin{aligned} g_{tt} &= \left(\frac{a^2 \sin^2(\theta)}{(2m\alpha_\pm)^2} - \frac{1}{\Sigma} + \frac{a^2 r \sin^2(\theta)}{2m\alpha_\pm^2 \Sigma} \right) \Delta + \frac{a^2 \sin^2(\theta)}{\alpha_\pm^2 \Sigma} (r - \alpha_\pm)^2 \,, \\ g_{\varphi_\pm t} &= \frac{a \sin^2(\theta)}{2m\alpha_\pm \Sigma} \big((\Sigma + 2mr)\Delta + (2m)^2 r (r - \alpha_\pm) \big) \;; \end{aligned}$$

to avoid ambiguities, we emphasise that $g_{\varphi_\pm t} = g(\partial_{\varphi_\pm}, \partial_t)$. Recalling that $\Delta = (r - r_+)(r - r_-)$, one sees that the choice

$$\alpha_\pm = r_\pm \tag{1.6.30}$$

leads indeed to a zero of order one in g_{tt}, as desired. As a bonus one obtains a zero of order one in $g_{\varphi_\pm t}$, which will shortly be seen to be useful as well.

Remark 1.6.6 It is of interest to check smoothness of the transition formulae from the coordinates $(u, v, \theta, \varphi_\pm)$ to the coordinates (v, r, θ, ϕ) of (1.6.16)-(1.6.17), or to

(u, r, θ, ψ) of (1.6.25)-(1.6.26). For example, near $r = r_+$ we have

$$\begin{aligned}
\varphi_+ &= \varphi - \frac{a}{2mr_+}t = \psi + \int_{r_+}^{r} \frac{a}{\Delta}dr - \frac{a}{2mr_+}\left(u + \int_{r_+}^{r} \frac{r^2+a^2}{\Delta}dr\right) \\
&= \psi - \frac{au}{2mr_+} + \frac{a}{2mr_+}\int_{r_+}^{r} \frac{2mr_+ - (r^2+a^2)}{\Delta}dr\,. \qquad (1.6.31)
\end{aligned}$$

Now,

$$2mr_+ - (r^2+a^2) = 2m(r_+ - r) - \Delta\,,$$

which vanishes at $r = r_+$. This shows that the integrand in (1.6.31) can be rewritten as a smooth function of r near $r = r_+$, and so the new angular coordinates φ_+ are smooth functions of (u, r, ψ) near $r = r_+$.

Similar calculations apply for φ_- near $r = r_-$, and for the coordinates (v, r, θ, ϕ). □

Keeping in mind (1.6.24), we see that (1.6.28) together with (1.6.30) is precisely what is needed for the Killing vectors $\partial_t + a(2mr_\pm)^{-1}\partial_\varphi$, tangent to the generators of the horizons $\{r = r_\pm\}$, to annihilate $\varphi_\pm$:

$$(\partial_t + \frac{a}{2mr_\pm}\partial_\varphi)\varphi_\pm = 0\,.$$

Thus $(\varphi_\pm, \theta)$ provide natural coordinates on the space of generators.

We can now get rid of the singularity in g_{rr} by introducing

$$u = t - f(r)\,, \quad v = t + f(r)\,, \quad f' = \frac{r^2+a^2}{\Delta}\,, \qquad (1.6.32)$$

so that, keeping in mind that $\Delta(r_\pm) = 0 \Longleftrightarrow r_\pm^2 + a^2 = 2mr_\pm$,

$$f(r) = \frac{2mr_\pm}{r_\pm - r_\mp}\ln|r - r_\pm| + h_\pm(r)\,,$$

where the $h_\pm$'s are smooth near $r = r_\pm$. This is somewhat similar to (1.2.44), but the function f has been chosen more carefully because of the θ–dependence of g_{rr}. One then has

$$dt = \frac{1}{2}(du + dv)\,, \quad dr = \frac{\Delta}{2(r^2+a^2)}(dv - du)\,,$$

so that

$$\begin{aligned}
g &= \frac{\Sigma\Delta}{4(r^2+a^2)^2}(du - dv)^2 + \frac{g_{tt}}{4}(du+dv)^2 + g_{\varphi_\pm t}d\varphi_\pm(du+dv) \\
&\quad + \left(r^2 + a^2 + \frac{2mar}{\Sigma}\right)\sin^2(\theta)d\varphi_\pm^2 + \Sigma d\theta^2\,.
\end{aligned}$$

There are no more unbounded terms in the metric, but one needs yet to get rid of a vanishing determinant: Indeed, as seen in (1.6.19), the determinant of the metric in the (r, t, θ, φ) variables equals $-\sin^2(\theta)\Sigma$. Since the Jacobian of the map $(t, r, \varphi) \mapsto (t, r, \varphi_\pm)$ is one, and that of the map $(t, r, \varphi_\pm) \mapsto (u, v, \varphi_\pm)$

is $-2f' = -2(r^2+a^2)/\Delta$, we find that the determinant of the metric in the $(u, v, \theta, \varphi_\pm)$ coordinates equals

$$-\frac{\Sigma\Delta^2\sin^2(\theta)}{4(r^2+a^2)^2}\,. \tag{1.6.33}$$

To get rid of this problem we set, as in (1.2.45),

$$\hat u = -\exp(-cu)\,, \quad \hat v = \exp(cv)\,. \tag{1.6.34}$$

Now,

$$du = -\frac{d\hat u}{c\hat u}\,, \quad dv = \frac{d\hat v}{c\hat v}\,,$$
$$\hat u\hat v = -\exp(c(v-u)) = -|r-r_\pm|^{4cmr_\pm/(r_\pm - r_\mp)}\exp(2ch_\pm(r))\,.$$

As before, one chooses

$$c = \frac{r_\pm - r_\mp}{4mr_\pm}\,,$$

so that, for $r > r_\pm$,

$$\hat u\hat v = -\exp(c(v-u)) = -(r-r_\pm)\exp\left(\frac{(r_\pm - r_\mp)h_\pm(r)}{2mr_\pm}\right)\,.$$

The functions

$$r \mapsto w := (r-r_\pm)\exp\left(\frac{(r_\pm - r_\mp)h_\pm(r)}{2mr_\pm}\right) \tag{1.6.35}$$

have a non-vanishing derivative at $r = r_\pm$. Hence, by the analytic implicit function theorem, there exist near $w = 0$ analytic functions $r_\pm(w)$ inverting (1.6.35). So, near $r = r_\pm$ we can write

$$r - r_\pm = -\hat u\hat v H_\pm(-\hat u\hat v)\,,$$

where the $H_\pm$'s are analytic near $\hat u\hat v = 0$, non-vanishing there, with a similar resulting formulae for Δ. Since g_{tt} and $g_{\varphi_\pm t}$ both contain a multiplicative factor $r - r_\pm \sim \hat u\hat v$, we conclude that the coefficients $g_{\varphi_\pm\hat u}$, $g_{\varphi_\pm\hat v}$, as well as

$$g_{\hat u\hat v} = -\frac{1}{c^2\hat u\hat v}g_{uv}$$

can be analytically extended across $r = r_\pm$. This is somewhat less obvious for

$$g_{\hat u\hat u} = \frac{1}{(c\hat u)^2}g_{uu}\,, \quad g_{\hat v\hat v} = \frac{1}{(c\hat v)^2}g_{vv}\,.$$

However, with some work one obtains

$$\begin{aligned} g_{uu} = g_{vv} &= \frac{\Delta\sin^2(\theta)}{(4mr_\pm)^2\Sigma}\left\{\frac{a^4\sin^2(\theta)}{(r^2+a^2)}\Big[-(\Delta+4mr)\Delta + (r_\pm^2 - r^2)\Big]\right. \\ &\qquad \left. +(\Delta+6mr)\Delta + 2a^2(r^2 - r_\pm^2)\right\}. \end{aligned}$$

This shows that both g_{uu} and g_{vv} have a zero of order two at $r = r_\pm$, which is precisely what is needed to cancel the singularities arising from $\hat{u}^{-2}$ in $g_{\hat{u}\hat{u}}$ and from $\hat{v}^{-2}$ in $g_{\hat{v}\hat{v}}$.

The Jacobian of the map $(u, v) \mapsto (\hat{u}, \hat{v})$ equals $c^2\hat{u}\hat{v}$, and it follows from (1.6.33) that the metric has Lorentzian signature in the coordinates $(\hat{u}, \hat{v}, \theta, \varphi_\pm)$.

The metric induced on the Boyer-Lindquist sections of the event horizons of the Kerr metric, as well as on the bifurcate Killing horizon, reads

$$ds^2 = (R^2 + a^2\cos^2(\theta))\,d\theta^2 + \frac{(R^2 + a^2)^2\sin^2(\theta)}{R^2 + a^2\cos^2(\theta)}d\varphi^2\,, \tag{1.6.36}$$

where $R = m \pm \sqrt{m^2 - a^2}$. We note that its Ricci scalar, which we denote by K, is [181]

$$K = \frac{(R^2 + a^2)(3a^2\cos^2(\theta) - R^2)}{(R^2 + a^2\cos^2(\theta))^3}\,.$$

1.6.2 Surface gravity, thermodynamical identities

Recall that the surface gravity κ_* of a Killing horizon $\mathscr{H}_*$ is defined through the formula

$$\partial_\mu(X^\alpha X_\alpha)|_{\mathscr{H}_*} = -2\kappa_* X_\mu\,. \tag{1.6.37}$$

The following provides a convenient procedure to calculate κ_*: Let b be any one-form which extends smoothly across the horizon and such that $b(X) = 1$. Then κ_* can be obtained from the equation

$$-2\kappa_* = -2\kappa_* b(X) = b(\nabla(X^\alpha X_\alpha))|_{\mathscr{H}_*}\,.$$

Note that the leftermost side of the last equation is independent of the choice of b, and so is therefore the right-hand side.

In order to implement this for the Kerr metric, recall that the Killing vector

$$X_* := \partial_t + \frac{a}{2mr_*}\partial_\varphi \equiv \partial_t + \frac{a}{a^2 + r_*^2}\partial_\varphi =: \partial_t + \Omega_*\partial_\varphi \tag{1.6.38}$$

is null on the Killing horizon $\mathscr{H}_* = \{r = r_*\}$, where $r_* \in \{r_-, r_+\}$ is one of the roots of Δ. As already pointed out, the parameter Ω_* is called the *angular velocity* of the horizon. The equation $g(X_*, X_*)|_{r=r_*} = 0$ is most easily checked using the following rewriting of the metric:

$$\begin{aligned} g &= \Sigma\left(\frac{1}{\Delta}dr^2 + d\theta^2\right) + \frac{\sin^2(\theta)}{\Sigma}\left(adt - (r^2 + a^2)d\varphi\right)^2 \\ &\quad -\frac{\Delta}{\Sigma}\left(dt - a\sin^2(\theta)\,d\varphi\right)^2\,. \end{aligned} \tag{1.6.39}$$

Let us use the coordinates (1.6.16)-(1.6.17), so that

$$dv = dt + \frac{r^2 + a^2}{\Delta}dr\,, \tag{1.6.40}$$

$$d\phi = d\varphi - \frac{a}{\Delta}dr\,. \tag{1.6.41}$$

Setting

$$b = dt + \frac{r^2 + a^2}{\Delta} dr = dv \,,$$

we see that b extends smoothly across the Killing horizon $\mathscr{H}_*$ and satisfies $b(X_*) = 1$. Thus, using

$$\begin{aligned} g^{rr} &= \frac{\Delta}{r^2 + a^2 \cos^2\theta} \\ g(X_*, X_*) &= -\frac{\Delta\left(r_*^2 + a^2\cos^2\theta\right)}{(a^2 + r_*^2)^2} + O((r - r_*)^2) \end{aligned} \tag{1.6.42}$$

(the last equation easily follows from the fact that the last term in the first line of (1.6.39) is $+O((r - r_*)^2)$) we find

$$\begin{aligned} \kappa_* &= -\frac{1}{2} b(\nabla(g(X_*, X_*))) = -\frac{1}{2}(dt + \frac{r^2 + a^2}{\Delta} dr)(g^{\nu\mu}\partial_\mu(g(X_*, X_*)\partial_\nu) \\ &= -\lim_{r \to r_*} \frac{(r^2 + a^2)}{2\Delta} g^{rr} \partial_r (g(X_*, X_*)) \\ &= \left.\frac{\partial_r \Delta}{2(r^2 + a^2)}\right|_{r = r_*} = \frac{r_* - m}{2mr_*} \\ &= \pm \frac{\sqrt{m^2 - a^2}}{2m(m \pm \sqrt{m^2 - a^2})} \,, \end{aligned} \tag{1.6.43}$$

where the plus sign applies to the *event horizon* $\{r = r_+\}$, and the minus sign should be used for the *Cauchy horizon* $\{r = r_-\}$.

In the extreme cases $m = \pm a$ only the plus sign is relevant. We see that κ vanishes then, and is not zero otherwise.

Let $J = ma$ be the "z-axis component" of the angular momentum vector, and let A_* be the area of the cross-sections of the event horizon: Denoting by $(x^A) = (\theta, \varphi)$ we have, using the fact that the metric is t-independent,

$$\begin{aligned} A_* &= \int_{r = r_*,\, v = \mathrm{const}} \sqrt{\det g_{AB}}\, d\theta\, d\varphi \\ &= \lim_{r \to r_*} \int_{r = \mathrm{const}',\, v = \mathrm{const}} \sqrt{\det g_{AB}}\, d\theta\, d\varphi \\ &= \lim_{r \to r_*} \int_{r = \mathrm{const}',\, t = \mathrm{const} + F(r)} \sqrt{\det g_{AB}}\, d\theta\, d\varphi \\ &= \lim_{r \to r_*} \int_{r = \mathrm{const}',\, t = \mathrm{const}} \sqrt{\det g_{AB}}\, d\theta\, d\varphi \\ &= 2\pi(r_*^2 + a^2) \int_0^\pi \sin(\theta)\, d\theta = 4\pi(r_*^2 + a^2) \,. \end{aligned} \tag{1.6.44}$$

By a direct calculation, or by general considerations [15, 98, 161, 272], one has the "thermodynamical identity"

$$\delta M_H = \frac{\kappa_*}{8\pi} \delta A_* + \Omega_* \delta J \,. \tag{1.6.45}$$

(Some care must be taken with the overall sign in (1.6.43) when the identity (1.6.45) is considered, as that sign is related to various orientations involved. The positive sign for the horizon $r_* = r_+$ is clearly consistent in this context.)

1.6.3 Carter's time machine

An intriguing feature of the Kerr metric in the region $\{r < 0\}$ is the existence of points at which

$$g_{\varphi\varphi} = g(\varphi, \varphi) < 0 \,. \tag{1.6.46}$$

In other words, there exists a non-empty region where the Killing vector ∂_φ is timelike. Indeed, we have

$$\begin{aligned} g_{\varphi\varphi} &= \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2(\theta)}{\Sigma} \sin^2(\theta) \\ &= \sin^2(\theta) \left(\frac{2a^2 m r \sin^2(\theta)}{a^2 \cos^2(\theta) + r^2} + a^2 + r^2 \right) \\ &= \frac{\sin^2(\theta)\left(a^4 + a^2 \cos(2\theta)\Delta + a^2 r(2m+3r) + 2r^4\right)}{a^2\cos(2\theta) + a^2 + 2r^2} \,. \end{aligned} \tag{1.6.47}$$

We are interested in the set where $g_{\varphi\varphi} < 0$. The second line above clearly shows that this never happens for $r \geq 0$, or for $|r|$ very large. Nevertheless, for all $m > 0$ the set

$$\begin{aligned} \mathscr{V} &:= \{g_{\varphi\varphi} < 0\} \\ &= \{r < 0 \,,\ \cos(2\theta) < \underbrace{-\frac{a^4 + 2a^2 m r + 3a^2 r^2 + 2r^4}{a^2 \Delta}}_{=:G(r)} \,, \\ &\qquad \Sigma \neq 0 \,,\ \sin(\theta) \neq 0\} \end{aligned} \tag{1.6.48}$$

is not empty. In order to see this, note that $G(0) = -1$, and $G'(0) = -4m/a^2 < 0$. This implies that for small negative r we have $G(r) > -1$, and hence there exists a range of θ near $\theta = \pi/2$ for which the inequality defining $\mathscr{V}$ is satisfied. This is illustrated in Figure 1.6.3.

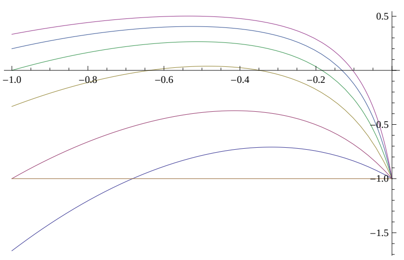

Figure 1.6.3: The function $G(ax)$ of (1.6.48) with $m/a \in \{0.5, 1, 2, 3, 4, 5\}$.

It turns out that *any two points within $\mathscr{V}$ can be connected by a future directed causal curve.* We show this in detail for points $p := (t, r, \theta, \varphi)$ and $p' := (t + T, r, \theta, \varphi)$ for any $T \in \mathbb{R}$: Indeed, for n large consider the curve

$$[0, 2n\pi] \ni s \mapsto \gamma(s) = (t + \frac{T}{2n\pi} s, r, \theta, \varphi \pm s) \,,$$

where the plus sign is chosen if ∂_φ is future-directed in $\mathscr{V}$, while the negative sign is chosen otherwise. Then $\dot\gamma = \pm\partial_\varphi + \frac{T}{2n\pi}\partial_t$, which is timelike future directed for all n large enough. As the φ coordinate is 2π-periodic, the curve γ starts at p and ends at p'.

A similar argument applies for general pairs of points within $\mathscr{V}$.

In particular the choice $T = 0$ and $n = 1$ gives a closed timelike curve.

Interestingly enough, the non-empty region $\mathscr{V}$ can be used to *connect any two points* p_1 *and* p_2 *lying in the region* $r < r_-$ *by a future-directed timelike curve.* To see this, choose some future-directed timelike curve γ_1 from p_1 to some point $p \in \mathscr{V}$, and some future-directed timelike curve from some point $p' \in \mathscr{V}$ to p_2. The existence of such curves γ_1 and γ_2 is easy to check, and follows e.g. by inspection of the projection diagram for the Kerr metric of Figure 4.7.3, p. 171 below. We can then connect p_1 with p_2 by a future-directed causal curve by first following γ_1 from p_1 to p, then a future-directed causal curve γ from p to p' lying in $\mathscr{V}$, and then following γ_2 from p' to p_2.

So, in fact, the region $\mathscr{V}$ provides a *time-machine* for the region $r < r_-$, a property which seems to have been first observed by Carter [42, 43].

We have been assuming that $m > 0$ in our discussion of the time-machine. It should, however, be clear from the arguments given that the time-travel mechanism for Kerr metrics just described exists in the region $r > 0$ if and only if $m < 0$.

1.6.4 Extreme case $a^2 = m^2$: horizon, near-horizon geometry, cylindrical ends

The coordinate transformation leading to (1.6.18) can be used for $a = m$ as well, leading to

$$\begin{aligned}\hat{g} &= -\Big(1-\frac{2mr}{\Sigma}\Big)dv^2 + 2drdv + \Sigma d\theta^2 - 2m\sin^2(\theta)d\phi dr \\ &\quad + \frac{(r^2+m^2)^2 - m^2\sin^2(\theta)\Delta}{\Sigma}\sin^2(\theta)d\phi^2 - \frac{4m^2 r\sin^2(\theta)}{\Sigma}d\phi dv\end{aligned} \tag{1.6.49}$$

As before it holds that

$$\det g = -\sin^2(\theta)\Sigma^2\,, \tag{1.6.50}$$

which shows that the metric is smooth and Lorentzian away from the set $\Sigma = 0$.

Near-horizon geometry

The near-horizon geometry g_{NHK} of the extreme Kerr solution can be obtained [14] by replacing the coordinates (t, r, φ) of (1.6.1), p. 63, by new coordinates $(\hat{t}, \hat{r}, \hat{\phi})$ defined as

$$r = m + \epsilon\hat{r}\,, \quad t = \epsilon^{-1}\hat{t}\,, \quad \varphi = \hat{\phi} + \frac{\hat{t}}{2m\epsilon}\,, \tag{1.6.51}$$

and passing to the limit $\epsilon \to 0$ compare Section 1.3.5. Some algebra leads to

$$g_{\mathrm{NHK}} = \frac{1+\cos^2(\theta)}{2}\Big[-\frac{\hat{r}^2}{r_0^2}\,d\hat{t}^2 + \frac{r_0^2}{\hat{r}^2}\,d\hat{r}^2 + r_0^2\,d\theta^2\Big] + \frac{2r_0^2\sin^2(\theta)}{1+\cos^2(\theta)}\Big(d\hat{\phi} + \frac{\hat{r}}{r_0^2}d\hat{t}\Big)^2\,, \tag{1.6.52}$$

where $r_0 = \sqrt{2}m$. This metric is singular at $\hat{r} = 0$, but a second change of coordinates

$$v = \hat{t} - \frac{r_0^2}{\hat{r}}\,, \qquad \tilde{\varphi} = \hat{\phi} - \log\left(\frac{\hat{r}}{r_0}\right)\,, \tag{1.6.53}$$

leads to a manifestly-regular form of the *near-horizon Kerr metric*:

$$g_{\rm NHK} = \frac{1+\cos^2(\theta)}{2}\Big[-\frac{\hat{r}^2}{r_0^2}\,dv^2 + 2\,dv\,d\hat{r} + r_0^2\,d\theta^2\Big] + \frac{2r_0^2\sin^2(\theta)}{1+\cos^2(\theta)}\Big(d\tilde{\varphi} + \frac{\hat{r}}{r_0^2}dv\Big)^2\,. \tag{1.6.54}$$

This is again a vacuum solution of the Einstein equations with a degenerate horizon located at $\hat{r} = 0$, but with rather a different asymptotic behaviour as the radial variable $\hat{r}$ tends to infinity.

Cylindrical ends

It turns out that the degenerate Kerr spacetimes contain CMC slices with *asymptotically conformally cylindrical* ends, in a sense which will be made precise: In Boyer-Lindquist coordinates the *extreme Kerr metrics*, with $a^2 = m^2$, take the form, changing φ to its negative if necessary,

$$\begin{aligned} g &= -dt^2 + \frac{2mr}{r^2+m^2\cos^2(\theta)}(dt - m\sin^2(\theta)d\varphi)^2 + (r^2+m^2)\sin^2(\theta)d\varphi^2 \\ &\quad + \frac{r^2+m^2\cos^2(\theta)}{(r-m)^2}dr^2 + (r^2+m^2\cos^2(\theta))d\theta^2\,, \end{aligned} \tag{1.6.55}$$

The metric γ induced on the slices $t = \text{const}$ reads, keeping in mind that $r > m$,

$$\begin{aligned} \gamma &= \frac{r^2+m^2\cos^2(\theta)}{(r-m)^2}dr^2 + (r^2+m^2\cos^2(\theta))d\theta^2 \\ &\quad + \frac{(r^2+m^2)^2 - (r-m)^2m^2\sin^2(\theta)}{r^2+m^2\cos^2(\theta)}\sin^2(\theta)d\varphi^2\,. \end{aligned} \tag{1.6.56}$$

Introducing a new variable $x \in (-\infty, \infty)$ defined as

$$dx = -\frac{dr}{r-m} \quad \Longrightarrow \quad x = -\ln(r-m)\,,$$

so that x tends to infinity as r approaches m from above, the metric (1.6.56) exponentially approaches

$$\begin{aligned} \gamma &\to_{x\to\infty} m^2(1+\cos^2(\theta))dx^2 + \mathring{g} \\ &= m^2(1+\cos^2(\theta))\Big(dx^2 + \underbrace{d\theta^2 + \frac{4\sin^2(\theta)}{(1+\cos^2(\theta))^2}d\varphi^2}_{=:\mathring{h}}\Big)\,. \end{aligned} \tag{1.6.57}$$

We see that the space-metric γ is, asymptotically, conformal to the product metric $dx^2 + \mathring{h}$ on the cylinder $\mathbb{R} \times S^2$.

Let us mention that the limiting metric, as one recedes to infinity along the cylindrical end of the extreme Kerr metric, can also be obtained from the metric (1.6.36) on the bifurcation surface of the event horizon by setting $a = m$ there:

$$\mathring{g} = m^2\left((1+\cos^2(\theta))d\theta^2 + \frac{4\sin^2(\theta)}{1+\cos^2(\theta)}d\varphi^2\right). \qquad (1.6.58)$$

We note that the slices $t = \text{const}$ are maximal. This follows from the fact that the unit normal n to these slices takes the form $n = n^t\partial_t + n^\varphi\partial_\varphi$, so that

$$\begin{aligned} \mathrm{tr}_\gamma K &= \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}n^\mu) \\ &= \frac{1}{\sqrt{|\det g|}}\Big(\partial_t(\sqrt{|\det g|}n^t) + \partial_\varphi(\sqrt{|\det g|}n^\varphi)\Big) = 0\,. \end{aligned}$$

It then follows from the scalar constraint equation shows that $R \geq 0$.

When studying the Lichnerowicz equation for metrics of cylindrical type it is of interest to study the sign of the scalar curvature of the various metrics occurring in the problem at hand. Recall that the scalar curvature, say κ, of a metric of the form $d\theta^2 + e^{2f(\theta)}d\varphi^2$ equals

$$\kappa = -2(f'' + (f')^2)\,.$$

Hence the sphere part $\mathring{h}$ of the limiting conformal metric appearing in (1.6.57) has scalar curvature equal to

$$-\frac{4\cos(2\theta)}{(\cos^2\theta + 1)^2}\,,$$

which is negative on the northern hemisphere and positive on the southern one. Finally, the metric $\mathring{g}$ has scalar curvature

$$\kappa = \frac{2(3\cos^2(\theta)-1)}{m^2(1+\cos^2(\theta))^3}\,.$$

and the reader will note that κ changes sign as well.

1.6.5 The Ernst map for the Kerr metric

A key role for proving uniqueness of the Kerr black holes is a *harmonic map representation* of the field equations: here, to every stationary axisymmetric solution of the vacuum Einstein field equations $(\mathscr{M}, g)$ one associates a pair of functions (f, ω), where f is norm of the axisymmetric Killing vector, say η:

$$f = g(\eta, \eta)\,,$$

while the function ω, called the *twist potential*, is defined as follows: One introduces, first, the *twist form* $\omega_\mu dx^\mu$ via the equation

$$\omega_\mu = \epsilon_{\mu\alpha\beta\gamma}\eta^\alpha\nabla^\beta\eta^\gamma\,.$$

It follows from the vacuum field equations that ω is closed, see (1.3.38), p. 46. So if, e.g., $\mathscr{M}$ is simply connected, there exists a function ω such that

$$\omega_\mu = \partial_\mu\omega\,.$$

The complex valued function $f + \mathrm{i}\omega$ is called the *Ernst potential.*

In Boyer-Lindquist coordinates of (1.6.6) the twist potential ω reads [103]

$$\omega = ma(\cos^3\theta - 3\cos\theta) - \frac{ma^3\cos\theta\sin^4\theta}{\Sigma}\,. \tag{1.6.59}$$

It is important for the study of such metrics that the leading order term in ω is uniquely determined by ma. The Ernst potential $f + \mathrm{i}\omega$ can now be obtained by reading $f = g_{\varphi\varphi}$ from (1.6.6).

1.6.6 The orbit space metric

Let M denote the space of orbits of the isometry group in the domain of outer communications:

$$M := \langle\langle \mathscr{M} \rangle\rangle / (\mathbb{R} \times U(1))\,.$$

Note that M can be viewed as the submanifold $\{t = 0 = \varphi\}$ of $\langle\langle \mathscr{M} \rangle\rangle$, coordinatized by $\tilde\theta \in [0, \pi]$ and $\tilde r$, with $r_+ < \tilde r < \infty$.

Let $X_1 = \partial_t$, $X_2 = \eta = \partial_\varphi$, and let

$$\mathscr{A} := \{\eta = 0\} = \{\tilde\theta = 0\} \cup \{\tilde\theta = \pi\}$$

be the *axis of rotation.* We will use the same symbol $\mathscr{A}$ for the axis of rotation in $\mathscr{M}$, as well as for the corresponding set in M. . The orbit space metric h on $M \setminus \mathscr{A}$ is defined as follows: for $Y, Z \in T(M \setminus \mathscr{A})$,

$$h(Y, Z) = g(Y, Z) - g^{ab} g(X_a, Y) g(X_b, Z)\,, \tag{1.6.60}$$

where g^{ab} is the matrix inverse to $g(X_a, X_b)$. Note that $\det g(X_a, X_b) < 0$ on $\langle\langle \mathscr{M} \rangle\rangle \setminus \mathscr{A}$, which shows that h is well defined there.

Since $g_{\tilde\theta t} = g_{\tilde\theta\varphi} = g_{\tilde r t} = g_{\tilde r\varphi} = 0$, h is obtained by simply forgetting the part of the metric involving dt and $d\varphi$:

$$h = (\tilde r^2 + a^2\cos^2\tilde\theta)\Big(\frac{d\tilde r^2}{(\tilde r - r_+)(\tilde r - r_-)} + d\tilde\theta^2\Big)\,. \tag{1.6.61}$$

So, $\{\tilde\theta = 0\}$ and $\{\tilde\theta = \pi\}$ are clearly smooth boundaries at finite distance for h, with h extending smoothly by continuity there. On the other hand, the nature of $\{\tilde r = r_+\}$ depends upon whether or not $r_+ = r_-$. In the subextreme case, where r_- and r_+ are distinct, the set $\{\tilde r = r_+\}$ is seen to be a totally geodesic boundary at finite distance by introducing a new coordinate x by the formula

$$\frac{dx}{d\tilde r} = \frac{1}{\sqrt{(\tilde r - r_+)(\tilde r - r_-)}}\,. \tag{1.6.62}$$

We then have

$$x_+ := \lim_{r \to r_+} x > -\infty\,,$$

as the right-hand side of (1.6.62) is integrable in $\tilde r$ near $\{\tilde r = r_+\}$. The same formula in the extreme case $\tilde r_- = \tilde r_+$ gives an x-variable which diverges logarithmically as r approaches r_-, leading to a *cylindrical end* for the metric h.

1.6.7 Kerr-Schild coordinates

The explicit, original, Kerr-Schild form of the Kerr metric (cf., e.g., [51]) reads

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2m\tilde{r}^3}{\tilde{r}^4 + a^2 z^2}\theta_\mu \theta_\nu \, ,$$

where

$$\theta_\mu dx^\mu = dx^0 - \frac{1}{\tilde{r}^2 + a^2}\left[\tilde{r}(xdx + ydy) + a(xdy - ydx)\right] - \frac{z}{\tilde{r}}dz \, ,$$

and where $\tilde{r}$ is defined implicitly as the solution of the equation

$$\tilde{r}^4 - \tilde{r}^2(x^2 + y^2 + z^2 - a^2) - a^2 z^2 = 0 \, .$$

This form of the metric makes manifest the asymptotic flatness of the metric, and turns out to be useful for performing *gluing constructions*, see [70].

1.6.8 Dain coordinates

Dain [103] has invented a system of coordinates which nicely exhibits the "Einstein-Rosen bridges" of the Kerr metric. One wants to write the space-part of the Kerr metric in the form

$$g = e^{-2\tilde{U}+2\alpha}\left(d\rho^2 + dz^2\right) + \rho^2 e^{-2\tilde{U}}\left(d\varphi + \rho B_\rho d\rho + A_z dz\right)^2 \, . \qquad (1.6.63)$$

If $|a| \le m$ let $r_+ = m + \sqrt{m^2 - a^2}$ be the largest root of Δ, and let $r_+ = 0$ otherwise. For

$$r > r_+ \, ,$$

so that $\Delta > 0$, define a new radial coordinate $\tilde{r}$ by

$$\tilde{r} = \frac{1}{2}\left(r - m + \sqrt{\Delta}\right) \, ; \qquad (1.6.64)$$

After setting

$$\rho = \tilde{r}\sin\tilde{\theta} \, , \quad z = \tilde{r}\cos\tilde{\theta} \, , \qquad (1.6.65)$$

one obtains (1.6.63). We have

$$r = \tilde{r} + m + \frac{m^2 - a^2}{4\tilde{r}} \, . \qquad (1.6.66)$$

We emphasize that while those coordinates bring the metric to the form (1.6.63), familiar in the context of the reduction of the stationary axi-symmetric vacuum Einstein equations to a harmonic map problem, the coordinate ρ in (1.6.65) is *not* the area coordinate needed for that reduction[14] *except* when $m = a$.

[14] The correct (ρ, z) coordinates for the harmonic map reduction are $\rho = \sqrt{\Delta}\sin(\theta)$, $z = (\tilde{r} - m)\cos\theta$. In the last coordinates the horizon lies on the axis $\rho = 0$, which is not the case for Dain's coordinates except if $a = m$.

To analyse the behavior near $r = 0$ we have to distinguish between the extreme and non-extreme cases. Let us first assume that $m^2 \neq a^2$. We can calculate $e^{\tilde{U}}$ from (1.6.6), and using (1.6.64) we then have

$$\tilde{U} = 2\ln\left(\frac{2\tilde{r}}{m}\right) - \ln\left|1 - \frac{a^2}{m^2}\right| + O(\tilde{r}). \tag{1.6.67}$$

With a little work it can now be seen that that $r = 0$ corresponds to another asymptotically flat region for the metric (1.6.63).

On the other hand, in the extreme case $m^2 = a^2$ one similarly finds

$$\tilde{U} = \ln\left(\frac{\tilde{r}}{2m}\right) + \frac{1}{2}\ln\left(1 + \cos^2(\theta)\right) + O(\tilde{r}). \tag{1.6.68}$$

This implies that the space geometry near $\tilde{r} = 0$ approaches is that of an "asymptotically cylindrical end", as discussed in general in Section 1.3.4.

1.7 Majumdar-Papapetrou multi black holes

In all examples discussed so far the black hole event horizon is a connected hypersurface in spacetime. In fact [37, 64, 75], there are no *regular, static, vacuum* solutions with several black holes, consistently with the intuition that gravity is an attractive force. However, static multi black holes become possible in presence of an electric field. Well-behaved examples are exhausted [94] by the *Majumdar-Papapetrou* black holes, in which the metric ${}^4g$ and the electromagnetic potential A take the form [197, 229]

$$ {}^4g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{1.7.1}$$
$$ A = u^{-1}dt\,, \tag{1.7.2}$$

with some nowhere vanishing function u. Einstein–Maxwell equations read then

$$\frac{\partial u}{\partial t} = 0\,, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,. \tag{1.7.3}$$

The solutions will be called *standard MP black holes* if the coordinates x^μ of (1.7.1)–(1.7.2) cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{\mu_i}{|\vec{x} - \vec{a}_i|}\,, \tag{1.7.4}$$

for some positive constants μ_i.

INCIDENTALLY: The property that these are the only regular black holes within the MP class has been proved in [84], see also [89, 146]; the fact that all multi-component regular static black holes are in the MP class has been established in [94], building upon the work in [200, 248, 255]; a gap in [94] related to analyticity of the metric has been removed in [75].

When $I = \infty$, it is a standard fact in potential theory that if the series (1.7.4) converges at some point, it converges to a smooth function everywhere away from the punctures. This case has been analysed in [61, Appendix B], where it was pointed out that the scalar $F_{\mu\nu}F^{\mu\nu}$ is unbounded whenever the $\vec{a}_i$'s have accumulation points. It follows from [84] that the case where $I = \infty$ and the $\vec{a}_i$'s do not have accumulation points cannot lead to regular asymptotically flat spacetimes.

Calculating the flux of the electric field on spheres $|\vec{x} - \vec{a}_i| = \epsilon \to 0$, one finds that μ_i is the electric charge carried by the puncture $\vec{x} = \vec{a}_i$: Indeed, let $F = dA$ be the Maxwell tensor, we have

$$F = -u^{-2} du \wedge dt = -u^{-2}\partial_\ell u \, dx^\ell \wedge dt\,.$$

The flux of F through a two-dimensional hypersurface S is defined as

$$\int_S \star F\,,$$

where $\star$ is the Hodge dual, see Appendix A.15, p. 274. A convenient orthonormal basis of T^*M is given by the co-frame

$$\theta^0 = u^{-1} dt\,, \quad \theta^\ell = u\, dx^\ell\,,$$

in terms of which we have

$$F = -u^{-2}\partial_\ell u \, \theta^\ell \wedge \theta^0\,.$$

This gives

$$\begin{aligned} \star F &= -u^{-2}\partial_\ell u \star (\theta^\ell \wedge \theta^0) = \frac{1}{2}\sum_{\ell jk} u^{-2}\partial_\ell u \, \epsilon^{\ell jk}\theta^j \wedge \theta^k \\ &= \frac{1}{2}\sum_{ijk} \partial_\ell u \epsilon^{\ell jk} dx^j \wedge dx^k\,. \end{aligned}$$

Consider a sphere $S(\vec{a}_i, \epsilon)$ of radius ϵ centred at $\vec{a}_i$, shifting the coordinates by $\vec{a}_i$ we can assume that $\vec{a}_i = \vec{0}$, then $\partial_\ell u$ approaches

$$-\mu_i \, \vec{x}/|\vec{x}|^3$$

on $S(\vec{a}_i, \epsilon)$ as ϵ tends to zero. Therefore

$$\lim_{\epsilon \to 0} \int_{S(\vec{a}_i, \epsilon)} \star F = -4\pi\mu_i\,.$$

In the current conventions the right-hand side is -4π times the charge, which establishes the claim.

We will see shortly that punctures correspond to connected components of the event horizon, so μ_i can be thought of as the negative of the electric charge of the i'th black hole.

Higher-dimensional generalisations of the MP solutions have been derived by Myers [213]. The metric and the electromagnetic potential take the form

$$^{n+1}g = -u^{-2} dt^2 + u^{\frac{2}{n-2}} \Big((dx^1)^2 + \ldots + (dx^n)^2 \Big)\,, \tag{1.7.5}$$

$$A = u^{-1} dt\,, \tag{1.7.6}$$

with u being time independent, and harmonic with respect to the flat metric $(dx^1)^2 + \ldots + (dx^n)^2$. Then, a natural candidate potential u for solutions with black holes takes the form

$$u = 1 + \sum_{i=1}^{N} \frac{\mu_i}{|\vec{x} - \vec{a}_i|^{n-2}} , \tag{1.7.7}$$

for some $\vec{a}_i \in \mathbb{R}^n$. Here configurations with $N = \infty$ and which are periodic in some variables are also of interest, as they could lead to Kaluza-Klein type four-dimensional solutions.

Let us point out some features of the geometries (1.7.5) with $N < \infty$. First, for large $|\vec{x}|$ we have

$$u = 1 + \frac{\sum_{i=1}^{N} \mu_i}{|\vec{x}|^{n-2}} + O(|\vec{x}|^{-(n-1)}) ,$$

so that the metric is asymptotically flat, with total ADM mass equal to $\sum_{i=1}^{N} \mu_i$.

Next, choose any i and denote by $r := |\vec{x} - \vec{a}_i|$ a radial coordinate centred at $\vec{a}_i$. Then the space-part g of the metric (1.7.5) takes the form

$$\begin{aligned} g &= u^{\frac{2}{n-2}} \Big((dx^1)^2 + \ldots + (dx^n)^2 \Big) = r^2 u^{\frac{2}{n-2}} \Big(\frac{dr^2}{r^2} + h \Big) \\ &= (r^{\frac{1}{n-2}} u)^{\frac{2}{n-2}} (d(\underbrace{\ln r}_{=:x})^2 + h) \\ &= (r^{\frac{1}{n-2}} u)^{\frac{2}{n-2}} (dx^2 + h) , \end{aligned} \tag{1.7.8}$$

where h is the unit round metric on S^{n-1}. Now, the metric $dx^2 + h$ is the canonical, complete, product metric on the cylinder $\mathbb{R} \times S^{n-1}$. Further

$$r^{\frac{1}{n-2}} u \to_{\vec{x} \to \vec{a}_i} \mu_i > 0 .$$

Therefore the space-part of the Majumdar-Papapetrou metric approaches a multiple of the canonical metric on the cylinder $\mathbb{R} \times S^{n-1}$ as $\vec{x}$ approaches $\vec{a}_i$. Hence, the space geometry is described by a complete metric which has one asymptotically flat region $|\vec{x}| \to \infty$ and N asymptotically cylindrical regions $\vec{x} \to \vec{a}_i$.

It has been shown by Hartle and Hawking [146] that, in dimension $n = 3$, every standard MP spacetime can be analytically extended to an electro–vacuum spacetime with I black hole regions. The calculation, which also provides some information in higher dimensions $n > 3$ but runs into difficulties there, proceeds as follows: Let, as before, $r = |\vec{x} - \vec{a}_i|$; for r small we replace t by a new coordinate v defined as

$$v = t + f(r) \qquad \Longrightarrow \qquad dt = dv - f'(r) dr ,$$

with a function f to be determined shortly. We obtain

$$\begin{aligned} {}^{n+1}g &= -u^{-2}(dv - f'dr)^2 + u^{\frac{2}{n-2}}(dr^2 + r^2 h) \\ &= -u^{-2} dv^2 + 2u^{-2} f' dv\, dr + \Big(u^{\frac{2}{n-2}} - u^{-2}(f')^2 \Big) dr^2 + u^{\frac{2}{n-2}} r^2 h . \end{aligned} \tag{1.7.9}$$

We have already seen that the last term $u^{\frac{2}{n-2}}r^2h$ is well behaved for r small. Let us show that in some cases we can choose f to get rid of the singularity in g_{rr}. For this we Taylor expand the non-singular part of u near $\vec{a}_i$ as follows:

$$u = \underbrace{\frac{\mu_i}{r^{n-2}} + 1 + \sum_{j\neq i}\frac{\mu_j}{|\vec{a}_j - \vec{a}_i|^{n-2}}}_{=:\mathring{u}} + r\hat{u} = \mathring{u}\Big(1 + O(r^{n-1})\Big)\,, \tag{1.7.10}$$

with $\hat{u}$ — an analytic function of r and of the angular variables, at least for small r. We choose f so that $\mathring{u}^{\frac{2}{n-2}} - \mathring{u}^{-2}(f')^2$ vanishes:

$$f' = \mathring{u}^{\frac{n-1}{n-2}}\,.$$

This shows that the function

$$^{n+1}g_{rr} = u^{\frac{2}{n-2}} - u^{-2}(f')^2 = \underbrace{\mathring{u}^{\frac{2}{n-2}}}_{\sim r^{-2}}\Big[\Big(\underbrace{\frac{u}{\mathring{u}}}_{1+O(r^{n-1})}\Big)^{\frac{2}{n-2}} - \Big(\underbrace{\frac{\mathring{u}}{u}}_{1+O(r^{n-1})}\Big)^2\Big] = O(r^{n-3})$$

is an analytic function of r and angular variables for small r.

The above works well when $n = 3$, in which case (1.7.9) reads

$$^{3+1}g \quad = \quad -\underbrace{u^{-2}}_{\sim r^2}dv^2 + 2\Big(\underbrace{\frac{\mathring{u}}{u}}_{=1+O(r^2)}\Big)^2 dv\,dr + \underbrace{g_{rr}}_{=O(1)}dr^2 + \underbrace{u^2r^2}_{=\mu_i^2+O(r)}h\,.$$

At $r = 0$ the determinant of ^{3+1}g equals $-\mu_i^4\det h \neq 0$, which implies that $^{3+1}g_{\mu\nu}$ can be analytically extended across the null hypersurface $\mathscr{H}_i := \{r = 0\}$ to a real-analytic Lorentzian metric defined in a neighborhood of $\mathscr{H}_i$. By analyticity the extended metric is vacuum. Obviously $\mathscr{H}_i$ is a Killing horizon for the Killing vector $\partial_t = \partial_v$, since $^{3+1}g_{vv}$ vanishes at $\mathscr{H}_i$.

We note that the differential of $g(\partial_v,\partial_v)$ vanishes at $r = 0$ as well, which shows that all horizons have vanishing surface gravity.

Let us return to general dimensions $n \geq 4$. The problem is that the determinant of the metric vanishes now at $r = 0$. One could hope that this can be repaired by a change of coordinates. For this, consider $^{n+1}g_{rv}$:

$$\begin{aligned}
^{n+1}g_{rv}dr\,dv \quad &= \quad u^{-2}f'dr\,dv = \Big(\frac{\mathring{u}}{u}\Big)^2\mathring{u}^{\frac{3-n}{n-2}}dr\,dv = \Big(1 + O(r^{n-2})\Big)\mu_i^{\frac{3-n}{n-2}}r^{n-3}dr\,dv \\
&= \quad \Big(1 + O(r^{n-2})\Big)\frac{\mu_i^{\frac{3-n}{n-2}}}{n-2}d(\underbrace{r^{n-2}}_{=:\rho})\,dv
\end{aligned}$$

We see that this term will give a non-vanishing contribution to the determinant if we introduce a new radial variable $\rho = r^{n-2}$. This, however, will wreak havoc in $^{n+1}g_{rr}dr^2$, as well as in various other terms because then $r = \rho^{\frac{1}{n-2}}$, which introduces fractional powers of the new coordinate ρ in the metric, leading to a continuous but non-manifestly-differentiable extension.

Now, none of these problems occur if $N = 1$, in which case $u = \mathring{u}$, hence ${}^{n+1}g_{rr} \equiv 0$; furthermore,

$$ {}^{n+1}g_{vv} = \mathring{u}^{-2} = \left(1 + \frac{\mu_i}{\rho}\right)^{-2} = \frac{\rho^2}{(\mu_i+\rho)^2}\,, \qquad (1.7.11)$$

$$ u^{\frac{2}{n-2}} r^2 = (\mathring{u}\rho)^{\frac{2}{n-2}} = (\mu_i + \rho)^{\frac{2}{n-2}}\,, \qquad (1.7.12)$$

$$ {}^{n+1}g_{rv} dr\, dv = \frac{\mathring{u}^{\frac{3-n}{n-2}}}{(n-2)r^{n-3}} d\rho dv = \frac{(\mathring{u}r^{n-2})^{\frac{3-n}{n-2}}}{(n-2)} d\rho dv = \frac{(\mu_i+\rho)^{\frac{3-n}{n-2}}}{(n-2)} d\rho dv\,, \quad (1.7.13)$$

which proves that the metric can be extended analytically across a Killing horizon $\{\rho = 0\}$, as desired. (The case $N = 1$ is of course spherically symmetric, so this calculation is actually a special case of that in Remark 1.2.12, p. 26.)

Equation (1.7.11) shows that the Killing vector $\partial_t = \partial_v$ is spacelike everywhere except at the horizon $\rho = 0$: $g(\partial_v, \partial_v) \geq 0$. In particular $g(\partial_v, \partial_v) \geq 0$ attains a minimum on the horizon, hence its derivative vanishes there. As before we conclude that the black hole is degenerate, $\kappa = 0$.

For $n \geq 4$ and $N > 1$ the above construction (or some slight variation thereof, with f not necessarily radial, chosen to obtain ${}^{n+1}g_{rr} = 0$) produces a metric which can at best be extended by continuity across a Killing horizon "located at $\vec{x} = \vec{a}_i$", but the extensions so obtained do not appear to be differentiable. The optimal degree of differentiability that one can obtain does not seem to be known in general. As such, it has been shown in [276] that the metric cannot be extended smoothly when $n \geq 4$ and $N = 2$ or 3.

More can be said for axi-symmetric solutions [39]: In dimension $n = 5$, C^2 extensions for multi-component axi-symmetric configurations can be constructed, and it is argued that generic such solutions do not possess C^3 extensions. Examples are constructed where smooth extensions are possible for one central component, or for an infinity string of components. In dimension $n \geq 5$, C^1 extensions for multi-component axi-symmetric configurations can be constructed, and it is argued that generic such solutions do not possess C^2 extensions.

PROBLEM 1.7.2 Study, for $n \geq 4$, whether (1.7.7) can be corrected by a harmonic function to give a smooth event horizon. Alternatively, show that there are no regular static multi-component electro-vacuum black holes in higher dimensions. □

1.7.1 Adding bifurcation surfaces

When trying to prove results about spacetimes containing non–degenerate Killing horizon, it is extremely convenient to have a compact bifurcation surface at hand. For example, this hypothesis is made throughout the classification theory of static (non–degenerate) black holes (cf. [71, 72] and references therein). The problem is, that while we have good control of the geometry of the domain of outer communications, various unpleasant things can happen at its boundary. In particular, in [239] it has been shown that there might be an obstruction for the extendability of a domain of outer communications in such a way that the extension comprises a compact bifuraction surface. Nevertheless, as far as applications are concerned, it suffices to have the following: Given a spacetime (M, g) with a domain of outer communications $\langle\langle M_{\rm ext}\rangle\rangle$ and a non–degenerate Killing horizon, there exists a spacetime (M', g'), with a domain of outer communications $\langle\langle M'_{\rm ext}\rangle\rangle$ which is isometrically diffeomorphic to $\langle\langle M_{\rm ext}\rangle\rangle$, such that all non–degenerate Killing horizons in (M', g') contain their bifurcation surfaces. Rácz and Wald have shown [239], under appropriate conditions, that this is indeed the case:

THEOREM 1.7.3 (I. Rácz & R. Wald, 96) *Let (M, g_{ab}) be a stationary, or stationary–rotating spacetime with Killing vector field X and with an asymptotically flat region $M_{\rm ext}$. Suppose that $J^+(M_{\rm ext})$ is globally hyperbolic with asymptotically flat Cauchy surface Σ which intersects the event horizon $\mathcal{N} = \partial\langle\langle M_{\rm ext}\rangle\rangle \cap J^+(M_{\rm ext})$ in a compact cross–section. Suppose that X is tangent to the generators of $\mathcal{N}$ and that the surface gravity of every connected component of $\mathcal{N}$ is a non–zero constant. Then there exists a spacetime (M', g'_{ab}) and an isometric embedding*

$$\Psi : \langle\langle M_{\rm ext}\rangle\rangle \to \langle\langle M'_{\rm ext}\rangle\rangle \subset M' \ ,$$

where $\langle\langle M'_{\rm ext}\rangle\rangle$ is a domain of outer communications in M', such that:

1. *There exists a one–parameter group of isometries of (M', g'_{ab}), such that the associated Killing vector field X' coincides with $\Psi^* X$ on $\langle\langle M'_{\rm ext}\rangle\rangle$.*
2. *Every connected component of $\partial\langle\langle M'_{\rm ext}\rangle\rangle$ is a Killing horizon which comprises a compact bifurcation surface.*
3. *There exists a local "wedge–reflection" isometry about every connected component of the bifurcation surface.*

It should be emphasized that neither field equations, nor energy inequalities, nor analyticity have been assumed above. However, constancy of surface gravity has been imposed; compare Section 1.3.3.

1.8 The Kerr-de Sitter/Kerr-anti-de Sitter metric

The Kerr-de Sitter (KdS) and the Kerr-anti de Sitter (KAdS) metrics are solutions of the vacuum Einstein equations with a cosmological constant [44]. They describe an axi-symmetric stationary black hole solving the vacuum Einstein equations with a positive (KdS) or negative (KAdS) cosmological constant. A description of some of their global properties can be found in [4, 45, 87]. Our presentation follows [223].

In *Boyer-Lindquist coordinates* the metric takes the form [44][15],

$$g \;=\; \rho^2\left(\frac{1}{\Delta_r}dr^2 + \frac{1}{\Delta_\theta}d\theta^2\right) + \frac{\sin^2(\theta)}{\rho^2\Xi^2}\Delta_\theta\left(adt - (r^2+a^2)d\varphi\right)^2$$

[15]The transformation between the coordinates used in [44] and the Boyer-Lindquist coordinates above is [45, p. 102]

$$\lambda = r\,, \qquad \mu = a\cos(\theta)\,, \qquad \psi = \tfrac{1}{a\Xi}\varphi\,, \qquad \chi + a^2\psi = \tfrac{1}{\Xi}t\,,$$
$$p = a^2\,, \qquad h = 1 - \tfrac{a^2\Lambda}{3}\,, \qquad e = 0\,, \qquad q = 0\,.$$

Note that the papers [44, 45] use the convention that *de Sitter* corresponds to $\Lambda < 0$; in other words, Carter's Λ is the negative of ours.

$$-\Delta_r \frac{1}{\rho^2 \Xi^2} \left(dt - a\sin^2(\theta)\, d\varphi\right)^2 , \tag{1.8.1}$$

where

$$\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2(\theta)\,, & (1.8.2)\\
\Delta_r &= (r^2+a^2)\left(1-\frac{\Lambda}{3}r^2\right) - 2mr\,, & (1.8.3)\\
\Delta_\theta &= 1+\frac{a^2\Lambda}{3}\cos^2(\theta)\,, & (1.8.4)\\
\Xi &= 1+\frac{a^2\Lambda}{3}\,, & (1.8.5)
\end{aligned}$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}$, and θ, φ being the standard coordinates parameterizing the sphere. Note that the metric functions are only well-defined away from zeros of ρ and Δ_r, and the determinant vanishes at $\sin(\theta) = 0$.

When $m = 0$, g is the *de Sitter* ($\Lambda > 0$; "dS") or the *anti-de Sitter* ($\Lambda < 0$; "AdS") metric: Indeed, for $a = 0$, and after obvious renaming of coordinates, one obtains directly the standard form of the (A)dS metric in manifestly static coordinates (T, R, Θ, Φ):

$$g_{\text{(A)dS}} = -(1-\frac{\Lambda R^2}{3})dT^2 + \frac{1}{1-\frac{\Lambda R^2}{3}}dR^2 + R^2\left(d\Theta^2 + \sin^2(\Theta)d\Phi^2\right)\,. \tag{1.8.6}$$

For $a \neq 0$ an explicit coordinate transformation which brings the metric to the form (1.8.6) has been given in [45, p. 102], see also [4, 149]:

$$\begin{aligned}
T &= \frac{t}{\Xi}\,,\\
R^2 &= \frac{1}{\Xi}\left(r^2\Delta_\theta + a^2\sin^2(\theta)\right)\,,\\
R\cos(\Theta) &= r\cos(\theta)\,,\\
\Phi &= \varphi - a\frac{\Lambda}{3\Xi}t\,. & (1.8.7)
\end{aligned}$$

If $a = 0 = \Lambda$ and $m \neq 0$ one obtains the Schwarzschild metric. In what follows we will assume that $\Lambda \neq 0$.

When $m \neq 0$ but $a = 0$ one obtains the Schwarzschild-de Sitter or the Schwarzschild-anti de Sitter metric:

$$g_{\text{S(A)dS}} = -(1-\frac{\Lambda R^2}{3}-\frac{2m}{R})dT^2 + \frac{1}{1-\frac{\Lambda R^2}{3}-\frac{2m}{R}}dR^2 + R^2\left(d\theta^2+\sin^2(\theta)d\varphi^2\right)\,. \tag{1.8.8}$$

The parameter m determines its mass

From now on we assume $ma \neq 0$. When $a < 0$, we can replace φ by $-\varphi$ to obtain a positive value of a, and therefore, to reduce the number of cases to be considered, we will assume

$$a > 0\,. \tag{1.8.9}$$

With the same reasoning we require

$$m > 0\,;$$

for $m < 0$ a new positive value for m is obtained by replacing r by $-r$.

The determinant of (1.8.1) is

$$\det(g) = -\frac{\rho^4}{\Xi^4}\sin^2(\theta) \tag{1.8.10}$$

and the metric is manifestly Lorentzian at $r = 0$, which shows that (1.8.1) defines a Lorentzian metric on any connected set on which the metric components remain bounded, i.e., away from zeros of Δ_r, the "ring singularity" at $\rho = 0$, and the trivial spherical coordinates singularity at $\theta \in \{0, \pi\}$.

The inverse metric reads

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu \;\; = \;\; & -\frac{\Xi^2}{\Delta_r\Delta_\theta\rho^2}\left(\left(a^2+r^2\right)^2\Delta_\theta - a^2\sin^2(\theta)\Delta_r\right)\partial_t^2 \\ & -2\frac{\Xi^2}{\Delta_r\Delta_\theta\rho^2}a\left(\left(a^2+r^2\right)\Delta_\theta - \Delta_r\right)\partial_t\partial_\varphi + \frac{\Delta_r}{\rho^2}\partial_r^2 \\ & + \frac{\Xi^2}{\Delta_r\Delta_\theta\rho^2\sin^2(\theta)}\left(\Delta_r - a^2\sin^2(\theta)\Delta_\theta\right)\partial_\varphi^2 + \frac{\Delta_\theta}{\rho^2}\partial_\theta^2\,. \end{aligned} \tag{1.8.11}$$

Note that

$$g^{tt} = \frac{g_{rr}\,g_{\theta\theta}\,g_{\varphi\varphi}}{\det(g)} = -\frac{\Xi^4}{\Delta_\theta}\times\frac{1}{\Delta_r}\times\frac{g_{\varphi\varphi}}{\sin^2(\theta)}$$

and

$$\mathrm{sgn}(g^{rr}) = \mathrm{sgn}(\Delta_r)\,,$$

so either r or $-r$ is a time-function when $\Delta_r < 0$, and t or $-t$ is a time-function when $\Delta_r > 0$ and $g_{\varphi\varphi} > 0$. In the region where ∂_φ is timelike,

$$\{(a^2+r^2)\Delta_\theta - a^2\Delta_r\sin^2(\theta) < 0\}\,,$$

which is nonempty for $\sin(\theta) \neq 0$, the orbits of the Killing vector ∂_φ are closed timelike curves.

The character of the principal orbits of the isometry group $\mathbb{R}\times U(1)$ is determined by the sign of the determinant

$$\det\begin{pmatrix} g_{tt} & g_{t\varphi} \\ g_{t\varphi} & g_{\varphi\varphi} \end{pmatrix} = -\frac{\Delta_r\Delta_\theta}{\Xi^4}\sin^2(\theta)\,. \tag{1.8.12}$$

Therefore, for $\theta = 0$ the orbits are either null or one-dimensional, while for $\theta \neq 0$ the orbits are timelike in the regions where $\Delta_r > 0$, spacelike where $\Delta_r < 0$ and null where $\Delta_r = 0$; the last case is only well-defined after the spacetime has been extended across the zeros of Δ_r, which then become Killing horizons.

Existence of well-behaved spacelike hypersurfaces requires

$$a^2\Lambda > -3\,, \tag{1.8.13}$$

which will be assumed in what follows.

The following identities are useful when studying the metrics (1.8.1):

$$\begin{aligned}
g_{\varphi\varphi} + a\sin^2(\theta) g_{t\varphi} &= \frac{\Delta_\theta\left(a^2+r^2\right)\sin^2(\theta)}{\Xi^2}\,,\\
g_{t\varphi} + a\sin^2(\theta) g_{tt} &= -\frac{a\Delta_\theta\sin^2(\theta)}{\Xi^2}\,,\\
a\; g_{\varphi\varphi} + \left(r^2+a^2\right) g_{t\varphi} &= \frac{a\Delta_r\sin^2(\theta)}{\Xi^2}\,,\\
a\; g_{t\varphi} + \left(r^2+a^2\right) g_{tt} &= -\frac{\Delta_r}{\Xi^2}\,,
\end{aligned}$$

as well as

$$g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\frac{\Delta_r\Delta_\theta\sin^2(\theta)}{\Xi^4}\,.$$

1.8.1 Asymptotic behavior

The K(A)dS metrics possess a boundary at infinity *à la Penrose*: Recall that a spacetime (M,g) admits a *conformal boundary at infinity* $\mathscr{I}$ if there exists a spacetime with non-empty boundary $(\tilde M,\tilde g)$ such that

1. M is the interior of $\tilde M$ and $\mathscr{I} = \partial\tilde M$, thus $\tilde M = M\cup\mathscr{I}$;
2. there exists $\Omega\in C^\infty(\tilde M)$ such that (a) $\tilde g = \Omega^2 g$ on M, (b) $\Omega>0$ on M, and (c) $\Omega=0$ and $d\Omega\neq 0$ on $\mathscr{I}$.

This applies to K(A)dS by choosing

$$\Omega = \sqrt{y^2}\,,\ \text{where } y := \frac{1}{r}\,.$$

Then

$$\begin{aligned}
\tilde g \;=\;& \Omega^2 g = y^2 g = -3\frac{1+a^2y^2\cos^2(\theta)}{-3a^2y^4+y^2\left(a^2\Lambda-3\right)+\Lambda+6my^3}dy^2\\
&+\frac{3a^2\Delta_\theta y^4\sin^2(\theta)-3a^2y^4+y^2\left(a^2\Lambda-3\right)+\Lambda+6my^3}{3\,\Xi^2\left(1+a^2y^2\cos^2(\theta)\right)}dt^2\\
&-2a\sin^2(\theta)\frac{a^2\Lambda y^2\left(a^2y^2+1\right)\cos^2(\theta)+a^2\Lambda y^2+\Lambda+6my^3}{3\,\Xi^2\left(1+a^2y^2\cos^2(\theta)\right)}dtd\varphi\\
&+\sin^2(\theta)\left(\frac{a^4\Lambda y^2+a^2y^2\cos^2(\theta)\left(3\,\Xi\left(a^2y^2+1\right)-6my\right)}{3\,\Xi^2\left(1+a^2y^2\cos^2(\theta)\right)}\right.\\
&\quad\left.+\frac{a^2\left(\Lambda+6my^3+3y^2\right)+3}{3\,\Xi^2\left(1+a^2y^2\cos^2(\theta)\right)}\right)d\varphi^2+\frac{1+a^2y^2\cos^2(\theta)}{\Delta_\theta}d\theta^2\,.
\end{aligned} \tag{1.8.14}$$

All the metric coefficients can now be analytically extended across, and to a neighborhood of, the set $\mathscr{I} := \{y = 0\}$. At $y = 0$ we have

$$\lim_{y\to 0} y^2 g = -\frac{3}{\Lambda} dy^2 + \frac{\Delta_\theta \sin^2(\theta)}{\Xi^2} d\varphi^2 + \frac{\Lambda}{3\Xi^2}\left(dt - a\sin^2(\theta)d\varphi\right)^2 + \frac{1}{\Delta_\theta} d\theta^2\,, \tag{1.8.15}$$

which is manifestly Lorentzian there, and hence in neighborhood of $\{y = 0\}$.

As $\tilde g$ on $\partial\tilde M$ is $\begin{cases} \text{Riemannian}, & \text{for } \Lambda > 0; \\ \text{Lorentzian}, & \text{for } \Lambda < 0, \end{cases}$ the conformal boundary $\mathscr{I}$ is $\begin{cases} \text{spacelike}, & \text{if } \Lambda > 0; \\ \text{timelike}, & \text{if } \Lambda < 0. \end{cases}$

In [45, p.102] it is emphasized that the K(A)dS metrics are *asymptotically (A)dS*, in the sense that the metrics approach the (A)dS metric as r goes to infinity. This can be immediately inferred from (1.8.15), where it is seen that the metric at $y = 0$ does not depend upon m, and hence coincides there with the corresponding conformal rescaling of the (A)dS metric. An explicit construction proceeds through the Kerr-Schild coordinates: Following [4, 136], we use the transformation

$$\begin{aligned} d\tau &= \frac{1}{\Xi}dt + \frac{2mr}{\left(1-\frac{r^2\Lambda}{3}\right)\Delta_r}dr\,, \\ d\phi &= d\varphi - \frac{a\Lambda}{3\Xi}dt + \frac{2mr\,a}{(r^2+a^2)\Delta_r}dr \end{aligned} \tag{1.8.16}$$

to obtain

$$g_{\mathrm{K(A)dS}} = g_{\mathrm{(A)dS}} + \frac{2mr}{\rho^2}(k_\mu dx^\mu)^2\,, \tag{1.8.17}$$

with

$$\begin{aligned} g_{\mathrm{(A)dS}} &= -\frac{\left(1-\frac{r^2\Lambda}{3}\right)\Delta_\theta}{\Xi}d\tau^2 + \frac{\rho^2}{\left(1-\frac{r^2\Lambda}{3}\right)(r^2+a^2)}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &\quad + \frac{(r^2+a^2)\sin^2(\theta)}{\Xi}d\phi^2\,, \end{aligned} \tag{1.8.18}$$

$$k_\mu dx^\mu = \frac{\Delta_\theta}{\Xi}d\tau + \frac{\rho^2}{\left(1-\frac{r^2\Lambda}{3}\right)(r^2+a^2)}dr - \frac{a\sin^2(\theta)}{\Xi}d\phi\,.$$

The metric $g_{\mathrm{(A)dS}}$ is the (A)dS metric in unusual coordinates, which can be verified by using the coordinate transformation [45, 136]

$$R^2 = \frac{r^2\Delta_\theta + a^2\sin^2(\theta)}{\Xi}\,, \tag{1.8.19}$$

$$R^2\sin^2(\Theta) = \frac{r^2+a^2}{\Xi}\sin^2(\theta)\,, \tag{1.8.20}$$

$$T = \tau\,, \tag{1.8.21}$$

$$\Phi = \phi \tag{1.8.22}$$

between (1.8.18) and the (A)dS metric in static coordinates (1.8.6). The vector field k^μ is null for both g and $g_{(\mathrm{A})\mathrm{dS}}$, as seen by a direct calculation, and tangent to a null geodesic congruence, as noted in [136].

Chapter 2

Emparan-Reall "black rings"

An interesting class of black hole solutions of the $4+1$ dimensional stationary vacuum Einstein equations has been found by Emparan and Reall [112] (see also [68, 111, 113] and references therein for further studies of the Emparan-Reall metrics). The metrics are asymptotically Minkowskian in spacelike directions, with an ergosurface and an event horizon having $S^1 \times S^2$ cross-sections. (The "ring" terminology refers to the S^1 factor in $S^1 \times S^2$.) Our presentation is an expanded version of [112], with a somewhat different labeling of the contants appearing in the metric; furthermore, the gravitational coupling constant G from that reference has been set to one here.[1]

While the mathematical interest of the black ring solutions is clear, their physical relevance is much less so, because of numerical evidence for their instability [110, 117, 152].

The starting point of the analysis is the following metric:

$$\begin{aligned} g &= -\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{A}d\psi\right)^2 \\ &\quad + \frac{F(y)}{A^2(x-y)^2}\left[-F(x)\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2\right)\right. \\ &\quad \left. + F(y)\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right], \end{aligned} \tag{2.0.1}$$

where $A > 0$, ν, and ξ_F are constants, and

$$F(\xi) = 1 - \frac{\xi}{\xi_F}, \tag{2.0.2}$$

$$G(\xi) = \nu\xi^3 - \xi^2 + 1 = \nu(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3). \tag{2.0.3}$$

One can check, e.g. using the MATHEMATICA package xACT [199], that (2.0.1) solves the vacuum Einstein equations.[2] The constant ν is chosen to satisfy $0 < \nu < \nu_* = 2/3\sqrt{3}$. The upper bound is determined by the requirement that

[1] I am grateful to R. Emparan and H. Reall for allowing me to reproduce their figures.

[2] I wish to thank Alfonso Garciá-Parrado and José Maria Martín-García for carrying out the xACT calculation.

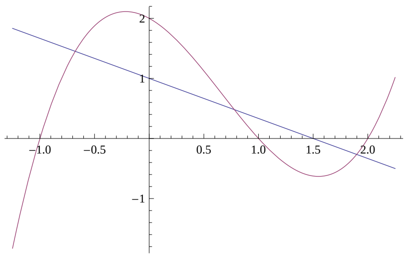

Figure 2.0.1: Representative plots of F and G.

the three roots $\xi_1 < \xi_2 < \xi_3$ of G are distinct and real. Note that $G(0) = 1$ so that $\xi_1 < 0$. Further $G' = 3\nu\xi^2 - 2\xi > 0$ for $\xi < 0$, which implies that $\xi_2 > 0$. Hence,

$$\xi_1 < 0 < \xi_2 < \xi_3 \,.$$

We will assume that[3]

$$\xi_2 < \xi_F < \xi_3 \,.$$

a definite choice of ξ_F consistent with this hypothesis will be made shortly. See Figure 2.0.1 for representative plots.

Requiring that

$$\xi_1 \le x \le \xi_2 \tag{2.0.4}$$

guarantees $G(x) \ge 0$ and $F(x) > 0$. On the other hand, both $G(y)$ and $F(y)$ will be allowed to change sign, as we will be working in the ranges

$$y \in (-\infty, \xi_1] \cup (\xi_F, \infty) \,. \tag{2.0.5}$$

INCIDENTALLY: Explicit formulae for the roots of G can be found, which are not particularly enlightening. For example, for $\nu \ge \nu_*$ one of the roots reads

$$\frac{\alpha}{6\nu} + \frac{2}{3\nu\alpha} + \frac{1}{3\nu} \,, \quad \text{where} \quad \alpha = \sqrt[3]{-108\,\nu^2 + 8 + 12\,\sqrt{3}\sqrt{27\,\nu^2 - 4}\nu} \,,$$

and a proper understanding of the various roots appearing in this equation also gives all solutions for $0 \le \nu < \nu_*$. Alternatively, in this last range of ν the roots belong to the set $\{(z_k + \frac{1}{2})\frac{2}{3\nu}\}_{k=0}^{2}$, with

$$z_k = \cos\left(\frac{1}{3}\Big[\arccos\Big(1 - \frac{27\nu^2}{2}\Big) + 2k\pi\Big]\right) .$$

□

Performing affine transformations of the coordinates, one can always achieve

$$\xi_1 = -1 \,, \quad \xi_2 = 1 \,,$$

but we will not impose these conditions in the calculations that follow.

[3] According to [112], the choice $\xi_F = \xi_2$ corresponds to the five-dimensional rotating black hole of [214], with one angular momentum parameter set to zero.

2.1 $x \in \{\xi_1, \xi_2\}$

There is a potential singularity of the $G^{-1}(x)dx^2 + G(x)F^{-1}(x)d\varphi^2$ terms in the metric at $x = \xi_1$, which can be handled as follows: consider, first, a metric of the form

$$h = \frac{dx^2}{x - x_0} + (x - x_0)f(x)d\varphi^2 \,, \quad f(x_0) > 0 \,. \tag{2.1.1}$$

Introducing

$$\tilde{\rho} = 2\sqrt{x - x_0} \,, \quad \varphi = \lambda\tilde{\varphi} \,, \tag{2.1.2}$$

one obtains

$$h = d\tilde{\rho}^2 + \frac{\lambda^2 f\left(x_0 + \frac{\tilde{\rho}^2}{4}\right)}{4}\tilde{\rho}^2 d\tilde{\varphi}^2 \,. \tag{2.1.3}$$

This defines a metric which smoothly extends through $\tilde{\rho} = 0$ (when f is smooth) if and only if $\tilde{\varphi}$ is periodically identified with period, say, 2π, and

$$\lambda = \frac{2}{\sqrt{f(x_0)}} \,. \tag{2.1.4}$$

Remark 2.1.1 In order to show that (2.1.4) implies regularity, set $x^1 = \tilde{\rho}\cos\tilde{\varphi}$, $x^2 = \tilde{\rho}\sin\tilde{\varphi}$, we then have

$$\begin{aligned} h &= \underbrace{d\tilde{\rho}^2 + \tilde{\rho}^2 d\tilde{\varphi}^2}_{\delta_{ab}dx^a dx^b} + \frac{\lambda^2\left(f\left(x_0 + \frac{\tilde{\rho}^2}{4}\right) - f(x_0)\right)}{4} \underbrace{\tilde{\rho}^2 d\tilde{\varphi}^2}_{\delta_{ab}dx^a dx^b - d\tilde{\rho}^2} \\ &= \delta_{ab}dx^a dx^b + \frac{\lambda^2\left(f\left(x_0 + \frac{\tilde{\rho}^2}{4}\right) - f(x_0)\right)}{4}\Big(\delta_{ab}dx^a dx^b - \tilde{\rho}^{-2}x^a x^b dx^a dx^b\Big) \,. \end{aligned}$$

As f is smooth, there exists a smooth function s such that

$$\frac{\lambda^2\left(f\left(x_0 + \frac{\tilde{\rho}^2}{4}\right) - f(x_0)\right)}{4} = \tilde{\rho}^2 s(\tilde{\rho}^2) \,,$$

so that

$$h = \left[\left(1 + s(\tilde{\rho}^2)\tilde{\rho}^2\right)\delta_{ab} - s(\tilde{\rho}^2)x^a x^b\right]dx^a dx^b \,, \tag{2.1.5}$$

which is manifestly smooth. This shows sufficiency of (2.1.4).

To show that (2.1.4) is necessary, note that from (2.1.3) we have $|D\tilde{\rho}|_h^2 = 1$. This implies that the integral curves of $D\tilde{\rho}$ are geodesics starting at $\{\tilde{\rho} = 0\}$. When $\{\tilde{\rho} = 0\}$ is a regular center one can run backwards a calculation in the spirit of the one that led to (2.1.5), using normal coordinates centered at $\tilde{\rho} = 0$ as a starting point, to conclude that the unit vectors orthogonal to the vector $\partial_{\tilde{\rho}}$ take the form $\pm\chi(\tilde{\rho})\partial_{\tilde{\varphi}}$, where $\chi(\tilde{\rho})^2\tilde{\rho}^2 \to_{\rho\to 0} 1$, and with $\partial_{\tilde{\varphi}}$ having periodic orbits with period 2π. Comparing with (2.1.3), (2.1.4) readily follows. □

In order to apply the above analysis to the last line of (2.0.1) at $x_0 = \xi_1$ we have

$$\begin{aligned} &\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2 = \\ &\quad = \frac{1}{\nu(x-\xi_2)(x-\xi_3)}\left(\frac{dx^2}{x-\xi_1} + \frac{\nu^2\xi_F(x-\xi_1)(x-\xi_2)^2(x-\xi_3)^2}{\xi_F - x}d\varphi^2\right) \\ &\quad = \frac{1}{\nu(x-\xi_2)(x-\xi_3)}\left(d\tilde{\rho}^2 + \frac{\lambda^2\nu^2\xi_F(x-\xi_2)^2(x-\xi_3)^2}{4(\xi_F - x)}\tilde{\rho}^2 d\tilde{\varphi}^2\right) , \end{aligned} \tag{2.1.6}$$

so that (2.1.4) becomes

$$\lambda = \frac{2\sqrt{\xi_F - \xi_1}}{\nu\sqrt{\xi_F}(\xi_2 - \xi_1)(\xi_3 - \xi_1)}. \tag{2.1.7}$$

For further purposes it is convenient to rewrite (2.1.6) as

$$\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2 = \frac{1}{H(x)}\left[d\tilde\rho^2 + \Big(1 + s(\tilde\rho^2)\tilde\rho^2\Big)\tilde\rho^2 d\tilde\varphi^2\right], \tag{2.1.8}$$

for a smooth function s with, of course,

$$H(\xi) = \nu(\xi - \xi_2)(\xi - \xi_3)\,. \tag{2.1.9}$$

When $\xi_F > \xi_2$ one can repeat this analysis at $x = \xi_2$, obtaining instead

$$\lambda = \frac{2\sqrt{\xi_F - \xi_2}}{\nu\sqrt{\xi_F}(\xi_2 - \xi_1)(\xi_3 - \xi_2)}. \tag{2.1.10}$$

Since the left-hand sides of (2.1.7) and (2.1.10) are equal, so must be the right-hand sides; their equality determines ξ_F:

$$\xi_F = \frac{\xi_1\xi_2 - {\xi_3}^2}{\xi_1 - 2\xi_3 + \xi_2} \tag{2.1.11}$$

(Elementary algebra shows that $\xi_2 < \xi_F < \xi_3$, as desired.) It should be clear that with this choice of ξ_F, for $y \neq \xi_1$, the (x, φ)–part of the metric (2.0.1) is a smooth (in fact, analytic) metric on S^2, with the coordinate x being the equivalent of the usual polar coordinate θ on S^2, except possibly at those points where the overall conformal factor vanishes or acquires zeros, which will be analysed shortly. Anticipating, the set obtained by varying x and φ and keeping $y = \xi_1$ will be viewed as S^2 with the north pole $x = \xi_1$ removed.

2.2 Signature

The calculation of the determinant of (2.0.1) reduces to that of a two-by-two determinant in the (t, ψ) variables, which equals

$$\frac{F^2(x)G(y)}{A^2(x-y)^2F(y)}\,, \tag{2.2.1}$$

leading to

$$\det g = -\frac{F^2(x)F^4(y)}{A^8(x-y)^8}\,, \tag{2.2.2}$$

so the signature is either $(-++++)$ or $(---++)$, except perhaps at the singular points $x = y$, or $F(x) = 0$ (which does not happen when $\xi_F > \xi_2$, compare (2.0.4)), or $F(y) = 0$.

Now, $F(x) > 0$, $G(x) > 0$ (away from the axes $x \in \{\xi_1, \xi_2\}$) thus, by inspection of (2.0.1), the signature is

$$\Big(\mathrm{sign}(-F(y)), \mathrm{sign}(-G(y)), \mathrm{sign}(-F(y)G(y)), +, +\Big)\,. \tag{2.2.3}$$

An examination of the four possible cases shows that a Lorentzian signature is obtained except if $F(y) > 0$ and $G(y) > 0$, which occurs for $y \in (\xi_1, \xi_2)$. So y's in this last range will not be of interest to us.

We start by considering

$$y \le \xi_1\,, \tag{2.2.4}$$

which leads to $F(y) > 0$ and $G(y) \le 0$.

2.3 $y = \xi_1$

Note that $G(\xi_1)$ vanishes; however, it should be clear from what has been said that $-(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2)$ is a smooth Riemannian metric if (ξ_1, ψ) are related to a new radial variable $\hat\rho$ and a new angular variable $\hat\varphi$ by

$$\hat\rho = 2\sqrt{\xi_1 - y} \in \mathbb{R}^+\,, \qquad \psi = \lambda\hat\varphi\,,$$

with λ given by (2.1.10) and $\hat\varphi$ being 2π-periodic. Analogously to (2.1.8), we thus have

$$-\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2\right) = \frac{1}{H(y)}\left[d\hat\rho^2 + \Big(1 + s(\hat\rho^2)\hat\rho^2\Big)\hat\rho^2 d\hat\varphi^2\right]. \tag{2.3.1}$$

Note that the remaining terms in (2.0.1) involving $d\psi$ are also well behaved: indeed, if we set $\hat x^1 = \hat\rho\cos\hat\varphi$, $\hat x^2 = \hat\rho\sin\hat\varphi$, then

$$(\xi_1 - y)d\psi = \frac{\lambda\hat\rho^2}{4}d\hat\varphi = \frac{\lambda}{4}(\hat x^1 d\hat x^2 - \hat x^2 d\hat x^1)\,,$$

which is again manifestly smooth.

2.4 Asymptotic flatness

We turn our attention now to the singularity $x = y$. Given our ranges of coordinates, this only occurs for $x = y = \xi_1$. So, at this stage, the coordinate t parameterises $\mathbb{R}$, the coordinates (y, ψ) are (related to polar) coordinates on $\mathbb{R}^2$, the coordinates (x, φ) are coordinates on S^2. If we think of $x = \xi_1$ as being the north pole of S^2, and we denote it by N, then g is an analytic metric on

$$\underbrace{\mathbb{R}}_{t} \times \Big(\big(\underbrace{\mathbb{R}^2}_{y,\psi \Leftrightarrow \hat\rho,\hat\varphi} \times \underbrace{S^2}_{x,\varphi \Leftrightarrow \tilde\rho,\tilde\varphi}\big) \setminus (\{0\} \times \{N\})\Big)\,.$$

Before passing to a detailed analysis of the metric for x and y close to ξ_1, it is encouraging to examine the leading order behavior of the last two lines in (2.0.1). Recall that (2.1.2) with $x_0 = \xi_1$ gives $x = \xi_1 + \tilde\rho^2/4$, and using (2.1.8) we rewrite the last line of (2.0.1), for small $\tilde\rho$,

$$\frac{F(y)^2}{A^2(x-y)^2}\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right) \approx \frac{F(\xi_1)^2}{A^2 H(\xi_1)(x-y)^2}\left(d\tilde\rho^2 + \tilde\rho^2 d\tilde\varphi^2\right)\,.$$

Similarly, with $y = \xi_1 - \hat\rho^2/4$, and with $\hat\rho$ small, the second line of (2.0.1) reads, keeping in mind (2.3.1),

$$-\frac{F(x)F(y)}{A^2(x-y)^2}\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2\right) \approx \frac{F(\xi_1)^2}{A^2H(\xi_1)(x-y)^2}\left(d\hat\rho^2 + \hat\rho^2 d\hat\varphi^2\right) .$$

Since $x - y = (\tilde\rho^2 + \hat\rho^2)/4$, adding one obtains

$$\frac{16F(\xi_1)^2}{A^2H(\xi_1)} \times \frac{1}{(\tilde\rho^2+\hat\rho^2)^2}\left(d\tilde\rho^2 + \tilde\rho^2 d\tilde\varphi^2 + d\hat\rho^2 + \hat\rho^2 d\hat\varphi^2\right) .$$

Up to an overall constant factor, this is a flat metric on $\mathbb{R}^4$, to which a *Kelvin inversion* $\vec x \mapsto \vec x/|\vec x|^2$ has been applied, rewritten using polar coordinates in two orthogonal planes.

We pass now to a complete analysis. Near the singular set $\mathbb{R}\times\{0\}\times\{N\}$, Emparan and Reall replace $(\tilde\rho, \hat\rho)$ by new radial variables $(\tilde r, \hat r)$ defined as

$$\tilde r = \frac{\tilde\rho}{B(\tilde\rho^2+\hat\rho^2)} , \quad \hat r = \frac{\hat\rho}{B(\tilde\rho^2+\hat\rho^2)} , \tag{2.4.1}$$

where B is a constant which will be determined shortly. This is inverted as

$$\tilde\rho = \frac{\tilde r}{B(\tilde r^2+\hat r^2)} , \quad \hat\rho = \frac{\hat r}{B(\tilde r^2+\hat r^2)} . \tag{2.4.2}$$

It is convenient to set

$$r = \sqrt{\tilde r^2 + \hat r^2} .$$

We note

$$x = \xi_1 + \frac{\tilde\rho^2}{4} = \xi_1 + \frac{\tilde r^2}{4B^2r^4} , \quad y = \xi_1 - \frac{\hat\rho^2}{4} = \xi_1 - \frac{\hat r^2}{4B^2r^4} ,$$
$$x - y = \frac{1}{4B^2r^2} .$$

This last equation shows that $x - y \to 0$ corresponds to $r \to \infty$.

Inserting (2.1.8) and (2.3.1) into (2.0.1) we obtain

$$\begin{aligned} g &= -\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{A}d\psi\right)^2 \\ &\quad + \frac{F(y)}{A^2(x-y)^2H(x)H(y)}\Bigg[F(x)H(x)\Big(d\hat\rho^2 + (1+s(\hat\rho^2)\hat\rho^2)\hat\rho^2 d\hat\varphi^2\Big) \\ &\quad + F(y)H(y)\Big(d\tilde\rho^2 + (1+s(\tilde\rho^2)\tilde\rho^2)\tilde\rho^2 d\tilde\varphi^2\Big)\Bigg] . \end{aligned} \tag{2.4.3}$$

The simplest terms arise from the first line above:

$$\begin{aligned} &-\frac{\xi_F - \xi_1 - \frac{\tilde r^2}{4B^2r^4}}{\xi_F - \xi_1 + \frac{\hat r^2}{4B^2r^4}}\left(dt + \lambda\sqrt{\frac{\nu}{\xi_F}}\frac{1}{4AB^2r^4}\hat r^2 d\hat\varphi\right)^2 \\ &\qquad = -\left(1 - \frac{1}{4(\xi_F-\xi_1)B^2r^2} + O(r^{-4})\right)\left(dt + O(r^{-4})\hat r^2 d\hat\varphi\right)^2 . \end{aligned} \tag{2.4.4}$$

In order to analyse the remaining terms, one needs to carefully keep track of all potentially singular terms in the metric: in particular, one needs to verify that the decay of the metric to the flat one is uniform with respect to directions, making sure that no problems arise near the rotation axes $\hat r = 0$ and $\tilde r = 0$. So we write the $\hat\varphi^2$ and the $\tilde\varphi^2$ terms from the last two lines of (2.4.3) as

$$\begin{aligned}
g_{\hat\varphi\hat\varphi}d\hat\varphi^2 + g_{\tilde\varphi\tilde\varphi}d\tilde\varphi^2 &= \frac{F(y)}{A^2(x-y)^2H(x)H(y)}\Big[F(x)H(x)\Big(1+s(\hat\rho^2)\hat\rho^2\Big)\hat\rho^2 d\hat\varphi^2 \\
&\quad +F(y)H(y)\Big(1+s(\tilde\rho^2)\tilde\rho^2\Big)\tilde\rho^2 d\tilde\varphi^2\Big] \\
&= \frac{16B^2F(y)}{A^2H(x)H(y)}\Big[F(x)H(x)\Big(1+O(r^{-4})\hat r^2\Big)\hat r^2 d\hat\varphi^2 \\
&\quad +F(y)H(y)\Big(1+O(r^{-4})\tilde r^2\Big)\tilde r^2 d\tilde\varphi^2\Big]. \qquad (2.4.5)
\end{aligned}$$

From

$$d\tilde\rho = \frac{1}{Br^4}\Big((\hat r^2-\tilde r^2)d\tilde r - 2\tilde r\hat r d\hat r\Big)\,, \quad d\hat\rho = \frac{1}{Br^4}\Big((\tilde r^2-\hat r^2)d\hat r - 2\tilde r\hat r d\tilde r\Big)\,,$$

one finds

$$\begin{aligned}
g_{\hat r\hat r} &= \frac{(4B)^2F(y)}{A^2H(x)H(y)r^4}\Big(F(x)H(x)(\hat r^2-\tilde r^2)^2 + 4F(y)H(y)\hat r^2\tilde r^2\Big) \\
&= \frac{(4B)^2F(y)}{A^2H(x)H(y)}\Big(F(x)H(x) + 4(F(y)H(y)-F(x)H(x))\frac{\hat r^2\tilde r^2}{r^4}\Big) \\
&= \frac{(4B)^2F(y)}{A^2H(x)H(y)}\Big(F(x)H(x) + O(r^{-4})\hat r^2\Big)\,, \qquad (2.4.6) \\
g_{\tilde r\tilde r} &= \frac{(4B)^2F(y)}{A^2H(x)H(y)r^4}\Big(F(y)H(y)(\hat r^2-\tilde r^2)^2 + 4F(x)H(x)\hat r^2\tilde r^2\Big) \\
&= \frac{(4B)^2F(y)}{A^2H(x)H(y)}\Big(F(y)H(y) + O(r^{-4})\tilde r^2\Big)\,, \qquad (2.4.7) \\
g_{\tilde r\hat r} &= \frac{2(4B)^2F(y)}{A^2H(x)H(y)r^4}\hat r\tilde r(\tilde r^2-\hat r^2)(F(y)H(y)-F(x)H(x)) \\
&= O(r^{-4})\hat r\tilde r\,. \qquad (2.4.8)
\end{aligned}$$

It is clearly convenient to choose B so that

$$\frac{(4B)^2F^2(\xi_1)}{A^2H(\xi_1)} = 1\,,$$

and with this choice (2.4.4)-(2.4.8) give

$$\begin{aligned}
g &= -\Big(1+O(r^{-2})\Big)\Big(dt + O(r^{-4})\hat r^2 d\hat\varphi\Big)^2 + O(r^{-4})\,\tilde r d\tilde r\,\hat r d\hat r \\
&\quad +\Big(1+O(r^{-2})\Big)\Big(d\hat r^2 + \hat r^2 d\hat\varphi^2\Big) + O(r^{-4})\hat r^4 d\hat\varphi^2 \\
&\quad +\Big(1+O(r^{-2})\Big)\Big(d\tilde r^2 + \tilde r^2 d\tilde\varphi^2\Big) + O(r^{-4})\tilde r^4 d\tilde\varphi^2\,. \qquad (2.4.9)
\end{aligned}$$

To obtain a manifestly asymptotically flat form one sets

$$\hat{y}^1 = \hat{r}\cos\hat{\varphi}\,,\ \hat{y}^2 = \hat{r}\sin\hat{\varphi}\,, \quad \tilde{y}^1 = \tilde{r}\cos\tilde{\varphi}\,,\ \tilde{y}^2 = \tilde{r}\sin\tilde{\varphi}\,,$$

then

$$\begin{aligned} \hat{r}d\hat{r} &= \hat{y}^1 d\hat{y}^1 + \hat{y}^2 d\hat{y}^2\,, \quad \hat{r}^2 d\hat{\varphi} = \hat{y}^1 d\hat{y}^2 - \hat{y}^2 d\hat{y}^1\,,\\ \tilde{r}d\tilde{r} &= \tilde{y}^1 d\tilde{y}^1 + \tilde{y}^2 d\tilde{y}^2\,, \quad \tilde{r}^2 d\tilde{\varphi} = \tilde{y}^1 d\tilde{y}^2 - \tilde{y}^2 d\tilde{y}^1\,, \end{aligned}$$

Introducing $(x^\mu) = (t, \hat{y}^1, \hat{y}^2, \tilde{y}^1, \tilde{y}^2)$, (2.4.9) gives indeed an asymptotically flat metric:

$$g = \Big(\eta_{\mu\nu} + O(r^{-2})\Big)dx^\mu dx^\nu\,.$$

2.5 $y \to \pm\infty$

In order to understand the geometry when $y \to -\infty$, one replaces y by

$$Y = -1/y\,.$$

Surprisingly, the metric can be analytically extended across $\{Y = 0\}$ to negative Y: indeed, we have

$$\begin{aligned} g \;=\;& -F(x)\left[\frac{dt^2}{F(y)} + 2\sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{AF(y)}dtd\psi\right.\\ &+\frac{1}{A^2}\Big(\frac{\nu(\xi_1-y)^2}{\xi_F - y} + \frac{G(y)}{(x-y)^2}\Big)d\psi^2 + \left.\frac{F(y)y^4}{A^2(x-y)^2G(y)}dY^2\right]\\ &+\frac{F^2(y)}{A^2(x-y)^2}\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\\ &\qquad\qquad \longrightarrow_{y\to-\infty}\\ &-F(x)\left[2\frac{\sqrt{\nu\xi_F}}{A}dtd\psi - \frac{2\nu\xi_1 + 2\nu x - 1 - \nu\xi_F}{A^2}d\psi^2 + \frac{1}{A^2\nu\xi_F}dY^2\right]\\ &+\frac{1}{A^2\xi_F^2}\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\Bigg]\,. \end{aligned} \tag{2.5.1}$$

Calculating directly, or using (2.2.2) and the transformation law for $\det g$, one has

$$\det g = -\frac{F^2(x)F^4(y)y^4}{A^8(x-y)^8} \longrightarrow_{y\to-\infty} -\frac{F^2(x)}{A^8\xi_F^4}\,, \tag{2.5.2}$$

which shows that the metric remains non-degenerate up to $\{Y = 0\}$. Further, one checks that all functions in (2.5.1) extend analytically to small negative Y; *e.g.*,

$$g(\partial_t, \partial_t) = g_{tt} = -\frac{F(x)}{F(y)} = -\frac{\xi_F - x}{\xi_F - y} = -\frac{(\xi_F - x)Y}{Y\xi_F + 1}\,, \tag{2.5.3}$$

etc.

To take advantage of the work done so far, in the region $Y < 0$ we replace Y by a new coordinate

$$z = -Y^{-1} > 0\,,$$

obtaining a metric which has the same form as (2.0.1):

$$
\begin{aligned}
g &= -\frac{F(x)}{F(z)}\left(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - z}{A}d\psi\right)^2 \\
&\quad + \frac{F(z)}{A^2(x-z)^2}\left[-F(x)\left(\frac{dz^2}{G(z)} + \frac{G(z)}{F(z)}d\psi^2\right)\right. \\
&\quad \left. + F(z)\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right]. \qquad (2.5.4)
\end{aligned}
$$

By continuity, or by (2.2.3), the signature remains Lorentzian, and (taking into account our previous analysis of the zeros of $G(x)$) the metric is manifestly regular in the range

$$\xi_3 < z < \infty\,. \qquad (2.5.5)$$

2.6 Ergoregion

Note that the "stationary" Killing vector

$$X := \partial_t\,,$$

which was timelike in the region $Y > 0$, is now spacelike in view of (2.5.3). In analogy with the Kerr solution, the part of the region where X is spacelike which lies outside of the black hole is called an *ergoregion*. (The fact that $\{\xi_x < z < \xi_F\}$ lies outside of the black hole region will be justified shortly.)

Since $g(X,X) = 0$ on the hypersurface $\{Y = 0\}$, this hypersurface is part of the boundary of the ergoregion, and the question arises whether or not this is a Killing horizon. Recall that, by definition, a Killing vector X is *normal* to its Killing horizon; in other words, it is orthogonal to every vector tangent to the Killing horizon (compare Appendix A.22). But, from (2.5.4) we find

$$
\begin{aligned}
g(\partial_t,\partial_\psi) &= -\sqrt{\frac{\nu}{\xi_F}}\frac{F(x)(\xi_1 - z)}{AF(z)} \\
&= -\sqrt{\frac{\nu}{\xi_F}}\frac{F(x)(\xi_1 - z)\xi_F}{A(\xi_F - z)} \\
&= -\sqrt{\frac{\nu}{\xi_F}}\frac{F(x)(\xi_1 + Y^{-1})\xi_F}{A(\xi_F + Y^{-1})} \\
&= -\sqrt{\frac{\nu}{\xi_F}}\frac{F(x)(\xi_1 Y + 1)\xi_F}{A(\xi_F Y + 1)} \to_{Y\to 0} -\sqrt{\frac{\nu}{\xi_F}}\frac{F(x)\xi_F}{A}\,.
\end{aligned}
$$

Since ∂_ψ is tangent to $\{Y = 0\}$, and since this last expression is not identically zero, we conclude that ∂_t is *not* normal to $\{Y = 0\}$. Hence $\{Y = 0\}$ is *not* a Killing horizons. Now, the part of the boundary of an ergoregion which lies outside the black hole is called *an ergosurface*. In the current case its topology is $S^1 \times S^2$: the factor S^1 corresponds to the rotations generated by ψ, and the factor S^2 corresponding to the spheres coordinatized by x and φ. Note that in the Kerr solution the ergosurface "touches" the event horizon at the axis of rotation, while here the event horizon and the ergosurface are separated by an open set.

2.7 Black ring

The metric (2.5.4) has a problem at $z = \xi_3$ because $G(\xi_3) = 0$. We have already shown how to solve that in regions where F was positive, but now $F(z) < 0$ so the previous analysis does not apply. Instead we replace ψ by a new (periodic) coordinate χ defined as

$$d\chi = d\psi + \frac{\sqrt{-F(z)}}{G(z)} dz\,. \tag{2.7.1}$$

However, this coordinate transformation wreaks havoc in the first line of (2.7.5). This is fixed if we replace t with a new coordinate v:

$$dv = dt + \sqrt{\tfrac{\nu}{\xi_F}}(z - \xi_1)\frac{\sqrt{-F(z)}}{AG(z)} dz\,. \tag{2.7.2}$$

INCIDENTALLY: The integrals above can be evaluated explicitly; for example, in (2.7.2) we have

$$\int \frac{\sqrt{z-\xi_F}}{(z-\xi_2)(z-\xi_3)} dz = \frac{\sqrt{\xi_3-\xi_F}}{\xi_3-\xi_2} \ln(z-\xi_3) + H(z), \tag{2.7.3}$$

where H is an analytic function defined in (ξ_F, ∞):

$$H(z) = \frac{2}{\xi_3-\xi_2}\left[\sqrt{\xi_F-\xi_2}\arctan\left(\sqrt{\frac{z-\xi_F}{\xi_F-\xi_2}}\right) - \sqrt{\xi_3-\xi_F}\ln\left(\sqrt{z-\xi_F}+\sqrt{\xi_3-\xi_F}\right)\right]. \tag{2.7.4}$$

□

In the (v, x, z, χ, ϕ)–coordinates the metric takes the form

$$\begin{aligned} ds^2 &= -\frac{F(x)}{F(z)}\left(dv - \sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A} d\chi\right)^2 \\ &\quad + \frac{1}{A^2(x-z)^2}\Bigg[F(x)\left(-G(z)d\chi^2 + 2\sqrt{-F(z)}d\chi dz\right) \\ &\quad + F(z)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\Bigg]. \end{aligned} \tag{2.7.5}$$

This is regular at

$$\mathscr{E} := \{z = \xi_3\}\,,$$

and the metric can be analytically continued into the region $\xi_F < z \le \xi_3$. One can check directly from (2.7.5) that $g(\nabla z, \nabla z)$ vanishes at $\mathscr{E}$. However, it is simplest to use (2.5.4) to obtain

$$g(\nabla z, \nabla z) = g^{zz} = -\frac{A^2(x-z)^2 G(z)}{F(x)F(z)} \tag{2.7.6}$$

in the region $\{z > \xi_3\}$, and to invoke analyticity to conclude that this equation remains valid on $\{z > \xi_F\}$. Equation (2.7.6) shows that $\mathscr{E}$ is a null hypersurface, with z being a time function on $\{z < \xi_3\}$, which is contained in a black hole region by the usual arguments (compare the paragraph around (1.6.21)).

We wish to show that $\{z = \xi_3\}$ is the event horizon: this will follow if we show that there is no event horizon enclosing the region $z < \xi_3$. For this, consider the "area function", defined as the determinant, say W, of the matrix

$$g(K_i, K_j)\,,$$

where the K_i's, $i = 1, 2, 3$, are the Killing vectors equal to ∂_t, ∂_ψ, and ∂_φ in the asymptotically flat region. In the original coordinates of (2.0.1) this equals

$$\frac{F(x)G(x)F(y)G(y)}{A^4(x-y)^4}\,, \tag{2.7.7}$$

with an identical expression where z replaces y in the coordinates of (2.5.4). By analyticity, or a direct calculation, this formula is not affected by the introduction of the coordinates of (2.7.5). Now,

$$F(y)G(y) = \frac{\nu}{\xi_F}(\xi_F - y)(y - \xi_1)(y - \xi_2)(y - \xi_3)\,,$$

and, in view of the range (2.0.4) of the variable x, the sign of (2.7.7) depends only upon the values of y and z. Since $F(y)G(y)$ behaves as $-\nu y^4$ for large y, W is negative both for $y < \xi_1$ and for $z > \xi_3$. Hence, at each point p of those two regions the set of vectors in $T_p\mathscr{M}$ spanned by the Killing vectors is timelike. So, suppose for contradiction, that the event horizon $\mathscr{H}$ intersects the region $\{y \in [-\infty, \xi_1) \cup z \in (\xi_3, \infty]\}$. Since $\mathscr{H}$ is a null hypersurface invariant under isometries, every Killing vector is tangent to $\mathscr{H}$. However, at each point at which W is negative there exists a linear combination of the Killing vectors which is timelike. This gives a contradiction because no timelike vector can be tangent to a null hypersurface.

We conclude that $\{z = \xi_3\}$ forms indeed the event horizon, with topology $\mathbb{R} \times S^1 \times S^2$: this is a "black ring".

2.8 Some further properties

It follows from (2.7.5) that the Killing vector field

$$\xi = \frac{\partial}{\partial v} + \frac{A\sqrt{\xi_F}}{\sqrt{\nu}(\xi_3 - \xi_1)}\frac{\partial}{\partial \chi} = \frac{\partial}{\partial t} + \frac{A\sqrt{\xi_F}}{\sqrt{\nu}(\xi_3 - \xi_1)}\frac{\partial}{\partial \psi} \tag{2.8.1}$$

is light-like at $\mathscr{E}$, which is therefore a *Killing horizon*. Equation (2.8.1) shows that the horizon is "rotating", with angular velocity

$$\Omega_{\mathscr{E}} = \frac{A\sqrt{\xi_F}}{\lambda(\xi_3 - \xi_1)\sqrt{\nu}} = \frac{A\sqrt{\nu}\xi_F(\xi_2 - \xi_1)}{2\sqrt{\xi_F - \xi_1}}\,; \tag{2.8.2}$$

recall that λ has been defined in (2.1.10). More precisely, in the coordinate system (v, χ, z, x, φ) the generators of the horizon are the curves

$$s \mapsto (v + s, \chi + \lambda\Omega_{\mathscr{E}} s, \xi_3, x, \varphi)\,.$$

We wish, next, to calculate the surface gravity of the Killing horizon $\mathscr{E}$. For this we start by noting that

$$\begin{aligned}
\xi^\flat &= g_{\mu\nu}\xi^\mu dx^\nu = g_{v\mu}dx^\nu + \lambda\Omega_{\mathscr{E}} g_{\chi\nu}dx^\nu \\
&= -\frac{F(x)}{F(z)}\left(1-\lambda\Omega_{\mathscr{E}}\sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A}\right)\left(dv-\sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A}d\chi\right) \\
&\quad +\frac{1}{A^2(x-z)^2}\lambda\Omega_{\mathscr{E}}F(x)\left(-G(z)d\chi+\sqrt{-F(z)}dz\right) \\
&= -\frac{F(x)(\xi_3-z)}{F(z)(\xi_3-\xi_1)}\left(dv-\sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A}d\chi\right) \\
&\quad +\frac{1}{A^2(x-z)^2}\lambda\Omega_{\mathscr{E}}F(x)\left(-G(z)d\chi+\sqrt{-F(z)}dz\right) \\
&= |_{z=\xi_3}\frac{\lambda\Omega_{\mathscr{E}}F(x)\sqrt{-F(\xi_3)}}{A^2(x-\xi_3)^2}dz\,, \qquad (2.8.3) \\
g(\xi,\xi) &= -\frac{F(x)}{F(z)}\left(1-\lambda\Omega_{\mathscr{E}}\sqrt{\frac{\nu}{\xi_F}}\frac{z-\xi_1}{A}\right)^2-\frac{\lambda^2\Omega_{\mathscr{E}}^2F(x)G(z)}{A^2(x-z)^2} \\
&= -\frac{F(x)(\xi_3-z)^2}{F(z)(\xi_3-\xi_1)^2}-\frac{\lambda^2\Omega_{\mathscr{E}}^2F(x)G(z)}{A^2(x-z)^2}\,, \\
d(g(\xi,\xi))|_{z=\xi_3} &= -\frac{\lambda^2\Omega_{\mathscr{E}}^2F(x)}{A^2(x-\xi_3)^2}G'(\xi_3)dz = -2\kappa\xi^\flat\,. \qquad (2.8.4)
\end{aligned}$$

Comparing (2.8.3) with (2.8.4) we conclude that

$$\kappa = \frac{\lambda\Omega_{\mathscr{E}}G'(\xi_3)}{2\sqrt{-F(\xi_3)}} = \frac{A\sqrt{\nu}}{2}\frac{\xi_F(\xi_3-\xi_2)}{\sqrt{\xi_3-\xi_F}}\,. \qquad (2.8.5)$$

Since $\kappa \neq 0$, one can further extend the spacetime obtained so far to one which contains a bifurcate Killing horizon, and a white hole region; we present the construction in Section 2.9 below. The global structure of the resulting spacetime resembles somewhat that of the Kruskal-Szekeres extension of the Schwarzschild solution.

The plot of Ω_H and κ (as well as some other quantities of geometric interest) in terms of ν can be found in Figure 2.8.1.

It is essential to understand the nature of the orbits of the isometry group, *e.g.* to make sure that the domain of outer communications does not contain any closed timelike curves. We have:

- The Killing vector ∂_t is timelike iff
$$F(y) > 0 \Longleftrightarrow y < \xi_F\,;$$
- The Killing vector ∂_φ is always spacelike;
- From (2.0.1) we have
$$g(\partial_\psi,\partial_\psi) = \frac{\nu F(x)(\xi_1-y)}{A^2(x-y)^2(\xi_F-y)}\times$$
$$\times\underbrace{\left((\xi_F-y)(\xi_2-y)(\xi_3-y)-(\xi_1-y)(x-y)^2\right)}_{(*)}\,. \qquad (2.8.6)$$

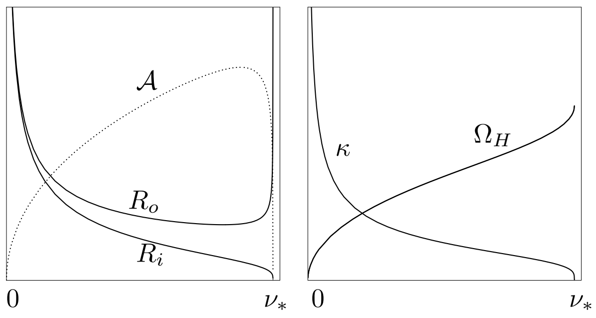

Figure 2.8.1: Plots, as functions of ν at fixed total mass m, of the radius of curvature R_i at $x = \xi_2$ of the S^1 factor of the horizon, the curvature radius R_o at $x = \xi_1$, total area $\mathcal{A}$ of the ring, surface gravity κ, and angular velocity at the horizon Ω_H. All quantities are rendered dimensionless by dividing by an appropriate power of m. Figure from [112].

For $y < \xi_1$ we can write

$$\underbrace{(\xi_F - y)}_{\geq (x-y)}\underbrace{(\xi_2 - y)}_{>(\xi_1 - y)}\underbrace{(\xi_3 - y)}_{>(x-y)} > (\xi_1 - y)(x-y)^2 \,,$$

which leads to $g_{\psi\psi} \geq 0$. Similarly, for $y > \xi_3$,

$$\underbrace{(y - \xi_F)}_{\leq (y-x)}\underbrace{(y - \xi_2)}_{<(y-\xi_1)}\underbrace{(y - \xi_3)}_{<(y-x)} < -(\xi_1 - y)(x-y)^2 \,,$$

so that ∂_ψ is spacelike or vanishing throughout the domain of outer communications.

- The metric induced on the level sets of t has the form

$$g_{yy}dy^2 + g_{\psi\psi}d\psi^2 + g_{xx}dx^2 + g_{\varphi\varphi}d\varphi^2 \,. \tag{2.8.7}$$

We have just seen that $g_{\psi\psi}$ is non-negative, and g_{xx} and $g_{\varphi\varphi}$ also are for x in the range (2.0.4). Further

$$g_{yy} = -\frac{F(x)F(y)}{A^2(x-y)^2G(y)} = \frac{F(x)}{A^2(x-y)^2\xi_F\nu} \times \frac{(y-\xi_F)}{(y-\xi_1)(y-\xi_2)(y-\xi_3)} \,,$$

an expression which is again non-negative in the ranges of interest. It follows that the hypersurfaces $\{t = \text{const}\}$ are spacelike.

- •2.8.1 The main topological features of the manifold $\mathscr{M}$ constructed so far •2.8.1: are summarised in Figure 2.8.3, see also Figure 2.8.2. One thus finds

$$\mathscr{M} = \mathbb{R} \times \left[\left(\mathbb{R}^2 \times S^2\right) \setminus \underbrace{(\vec{0}, N)}_{=:i^o}\right] ,$$

where $\vec{0}$ is the origin of $\mathbb{R}^2$, and N is the north pole of S^2, with the first $\mathbb{R}$ factor corresponding to time. The point i^o which has been removed from the $\mathbb{R}^2 \times S^2$ factor can be thought of as representing "spatial infinity".

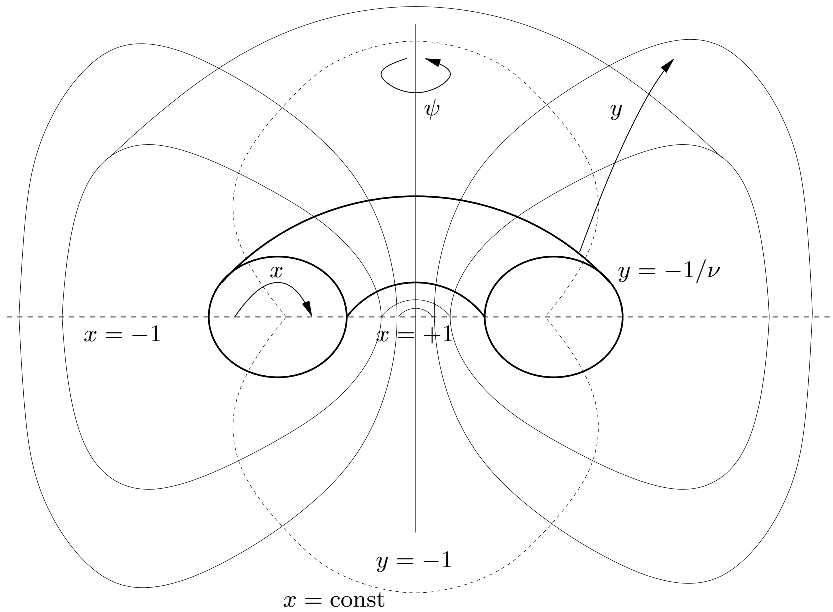

Figure 2.8.2: Coordinate system for black ring metrics, from [111]. The diagram sketches a section at constant t and φ. Surfaces of constant y are ring-shaped, while x is a polar coordinate on S^2. Infinity lies at $x = y = -1$.

INCIDENTALLY: The metric h induced on the sections of the horizon $\{v = \text{const}\,,\ z = \xi_3\}$ can be obtained from (2.8.7) by first neglecting the dy^2 terms, and then passing to the limit $y \to \xi_3$. (By general arguments, or by a direct calculation from (2.7.5), this coincides with the metric of the sections $\{v = \text{const}\}$ of the event horizon $\mathscr{E}$.) One finds

$$h = \frac{\lambda^2\nu(\xi_F - x)(\xi_3 - \xi_1)^2}{\xi_F A^2(\xi_3 - \xi_F)} d\hat{\varphi}^2 + \frac{F^2(\xi_3)}{A^2(x - \xi_3)^2}\left(\frac{dx^2}{G(x)} + \frac{\lambda^2 G(x)}{F(x)} d\tilde{\varphi}^2\right),$$

so that (recall (2.1.10))

$$\sqrt{\det h} = \frac{\lambda^2\nu^{1/2}(\xi_3 - \xi_F)^{3/2}(\xi_3 - \xi_1)}{\xi_F^2 A^3(x - \xi_3)^2} = \frac{4(\xi_3 - \xi_F)^{3/2}(\xi_F - \xi_1)}{A^3\nu^{3/2}\xi_F^3(\xi_3 - \xi_1)^2(\xi_2 - \xi_1)^2} \times \frac{1}{(x - \xi_3)^2}.$$

By integration in $x \in (\xi_1, \xi_2)$ and in the angular variables $\tilde{\varphi}, \hat{\varphi} \in (0, 2\pi)$ one obtains the area of the sections of the event horizon:

$$\mathcal{A} = \frac{16\pi^2}{A^3\nu^{3/2}} \frac{(\xi_3 - \xi_F)^{3/2}(\xi_F - \xi_1)}{\xi_F^3(\xi_3 - \xi_2)(\xi_2 - \xi_1)(\xi_3 - \xi_1)^2}. \tag{2.8.8}$$

□

If $\nu = \nu_*$ then the black ring and the black hole degenerate to the same solution with $\xi_2 = \xi_F = \xi_3$. This is the $\mu = a^2$ limit of the five-dimensional rotating black hole, for which the horizon disappears, and is replaced by a naked singularity.

The mass m and the angular momentum J can be calculated using Komar integrals:

$$m = \frac{3\pi}{2A^2} \frac{\xi_F - \xi_1}{\nu\xi_1^2(\xi_2 - \xi_1)(\xi_3 - \xi_1)}, \tag{2.8.9}$$

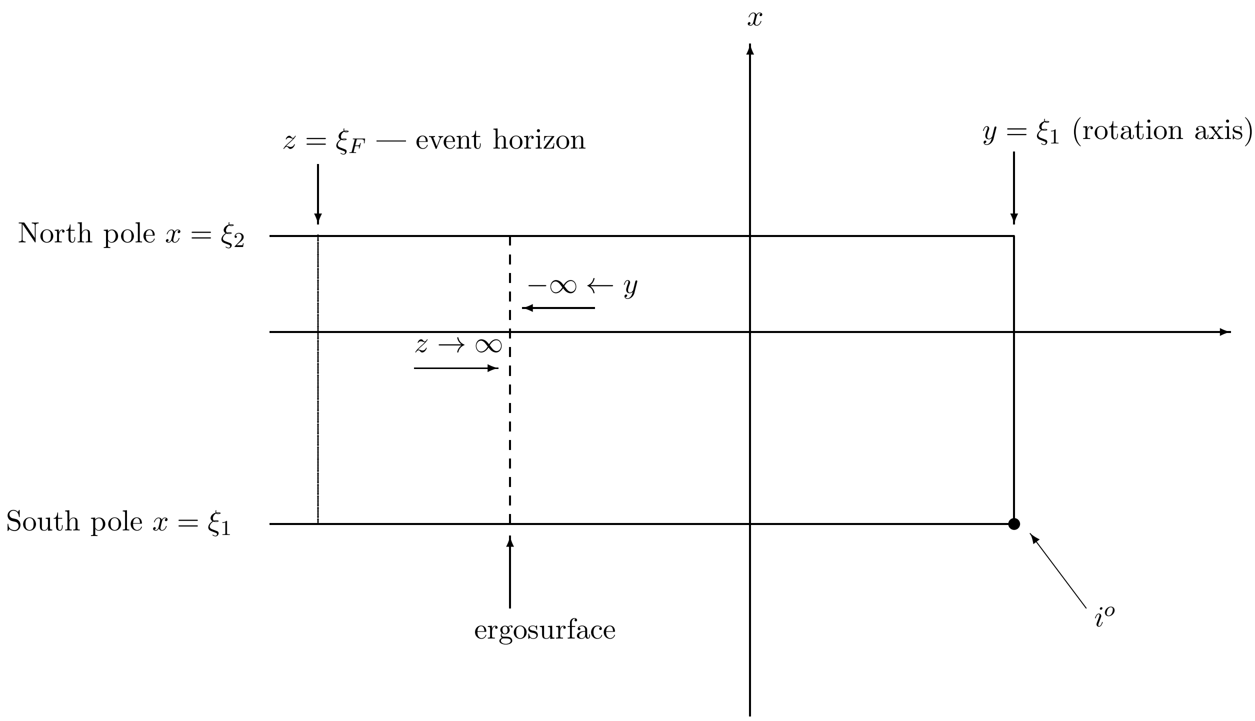

Figure 2.8.3: Space sections of the Emparan-Reall black holes, with the angular variables φ and ψ suppressed. The x variable runs along the vertical axis, the y variable runs along the horizontal axis to the right of the ergosurface, while the z coordinate is used horizontally to the left of the ergosurface. i^o is the point at infinity.

$$J = \frac{2\pi}{A^3} \frac{(\xi_F - \xi_1)^{5/2}}{\nu^{3/2} \xi_F^3 (\xi_2 - \xi_1)^2 (\xi_3 - \xi_1)^2} . \tag{2.8.10}$$

Thus, m and J are rather complicated functions of the independent parameters A and ν in view of (2.1.11).

Recall that the spin of the Myers-Perry five-dimensional black holes is bounded from above [214]:

$$\frac{J^2}{m^3} < \frac{32}{27\pi} . \tag{2.8.11}$$

The corresponding ratio for the solutions here is

$$\frac{J^2}{m^3} = \frac{32}{27\pi} \frac{(\xi_3 - \xi_1)^3}{(2\xi_3 - \xi_1 - \xi_2)^2 (\xi_2 - \xi_1)} . \tag{2.8.12}$$

These ratios are plotted as a function of ν in Figure 2.8.4. Rather surprisingly, this ratio is bounded *from below*:

$$\frac{J^2}{m^3} > 0.8437 \frac{32}{27\pi} . \tag{2.8.13}$$

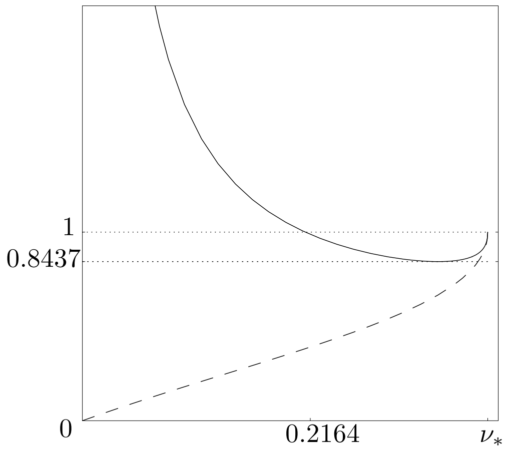

Figure 2.8.4: $(27\pi/32)J^2/m^3$ as a function of ν. The solid line corresponds to the Emparan-Reall solutions, the dashed line to the Myers-Perry black holes. The two dotted lines delimit the values for which both solutions with the same mass and spin exist. From [112].

For $0.2164 < \nu < \nu_*$, there are *two* black ring solutions with the same values of m and J (but different $\mathcal{A}$). Moreover, these satisfy the bound (2.8.11) so there is also a black hole with the same values of m and J. This implies that the uniqueness theorems valid in four dimensions do not have a simple generalisation to five dimensions, compare [154].

Some algebra shows that the quantities m, J, Ω_H, κ and $\mathcal{A}$ satisfy a *Smarr relation*

$$m = \frac{3}{2}\left(\frac{\kappa\mathcal{A}}{8\pi} + \Omega_H J\right). \tag{2.8.14}$$

2.9 A Kruskal-Szekeres type extension

So far we have seen how to construct an Eddington-Finkelstein type extension of the ER metric (2.0.1) across a Killing horizon. We will denote by $(\mathscr{M}_{I\cup II}, g)$ that extension. We will denote by $(\mathscr{M}_I, g)$ the subset of $(\mathscr{M}_{I\cup II}, g)$ exterior to the Killing horizon, see Figure 2.9.1, p. 114.

To construct a Kruskal-Szekeres type extension we follow [71] and consider, first, the form of the metric (2.5.4) with the coordinate z in the range $z \in (\xi_3, \infty)$. There we define new coordinates w, v by the formulae

$$dv = dt + \frac{bdz}{(z-\xi_3)(z-\xi_0)}\,, \qquad dw = dt - \frac{bdz}{(z-\xi_3)(z-\xi_0)}\,, \tag{2.9.1}$$

where b and ξ_0 are constants to be chosen shortly. Similarly to the construction of the extension of the Kerr metric in [32, 43], we define a new angular coordinate $\hat{\psi}$ by:

$$d\hat{\psi} = d\psi - adt\,, \tag{2.9.2}$$

where a is a constant to be chosen later. Let

$$\sigma := \frac{1}{A}\sqrt{\frac{\nu}{\xi_F}}\,. \tag{2.9.3}$$

Using (2.9.1)–(2.9.2), we obtain

$$dt = \tfrac{1}{2}(dv + dw)\,, \tag{2.9.4}$$

$$dz = \underbrace{\frac{(z-\xi_3)(z-\xi_0)}{2b}}_{=:H(z)/2}(dv - dw)\,, \quad d\psi = d\hat{\psi} + \tfrac{a}{2}(dv + dw)\,, \tag{2.9.5}$$

which leads to

$$g_{vv} = g_{ww} = -\frac{F(x)}{4F(z)}\Big(1 + a\sigma(\xi_1 - z)\Big)^2 - \frac{F(x)F(z)}{4A^2(x-z)^2}\left(\frac{a^2G(z)}{F(z)} + \frac{H^2(z)}{G(z)}\right)\,, \tag{2.9.6}$$

$$g_{vw} = -\frac{F(x)}{4F(z)}\Big(1 + a\sigma(\xi_1 - z)\Big)^2 - \frac{F(x)F(z)}{4A^2(x-z)^2}\left(\frac{a^2G(z)}{F(z)} - \frac{H^2(z)}{G(z)}\right)\,, \tag{2.9.7}$$

$$g_{v\hat{\psi}} = g_{w\hat{\psi}} = -\frac{F(x)}{2F(z)}\sigma(\xi_1 - z)\Big(1 + a\sigma(\xi_1 - z)\Big) - \frac{F(x)G(z)a}{2A^2(x-z)^2}\,, \tag{2.9.8}$$

$$g_{\hat{\psi}\hat{\psi}} = -\frac{F(x)}{F(z)}\sigma^2(\xi_1 - z)^2 - \frac{F(x)G(z)}{A^2(x-z)^2}\,. \tag{2.9.9}$$

The Jacobian of the coordinate transformation is

$$\frac{\partial(w, v, \hat{\psi}, x, \varphi)}{\partial(t, z, \psi, x, \varphi)} = 2\frac{\partial v}{\partial z} = \frac{2b}{(z-\xi_2)(z-\xi_3)}\,.$$

In the original coordinates (t, z, ψ, x, φ) the determinant of g was

$$\det(g_{(t,z,\psi,x,\varphi)}) = -\frac{F^2(x)F^4(z)}{A^8(x-z)^8}\,, \tag{2.9.10}$$

so that in the new coordinates it reads

$$\det(g_{(w,v,\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z-\xi_2)^2(z-\xi_3)^2}{4A^8b^2(x-z)^8}\,. \tag{2.9.11}$$

This last expression is negative on $(\xi_F, \infty) \setminus \{\xi_3\}$, and has a second order zero at $z = \xi_3$. In order to remove this degeneracy one introduces

$$\hat{v} = \exp(cv)\,, \quad \hat{w} = -\exp(-cw)\,, \tag{2.9.12}$$

where c is some constant to be chosen. Hence we have

$$d\hat{v} = c\hat{v}dv\,, \quad d\hat{w} = -c\hat{w}dw\,, \tag{2.9.13}$$

and the determinant in the coordinates $(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)$ reads

$$\det(g_{(\hat{w},\hat{v},\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z-\xi_2)^2(z-\xi_3)^2}{4A^8b^2(x-z)^8c^4\hat{v}^2\hat{w}^2}\,. \tag{2.9.14}$$

But one has $\hat{v}^2\hat{w}^2 = \exp(2c(v-w))$, so that

$$\begin{aligned}
\hat{v}^2\hat{w}^2 &= \exp\left(4cb\int \frac{1}{(z-\xi_2)(z-\xi_3)}dz\right) \\
&= \exp\left(\frac{4cb}{(\xi_3-\xi_2)}(\ln(z-\xi_3)-\ln(z-\xi_2))\right). \qquad (2.9.15)
\end{aligned}$$

Taking into account (2.9.15), and the determinant (2.9.14), we choose the constant c to satisfy:

$$\frac{2cb}{(\xi_3-\xi_2)} = 1\,. \qquad (2.9.16)$$

We obtain

$$\hat{v}\hat{w} = -\frac{z-\xi_3}{z-\xi_2}\,, \qquad (2.9.17)$$

and

$$\det(g_{(\hat{w},\hat{v},\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z-\xi_2)^4}{4A^8b^2(x-z)^8c^4}\,. \qquad (2.9.18)$$

With this choice, the determinant of g in the $(\hat{w},\hat{v},\hat{\psi},x,\varphi)$ coordinates extends to a strictly negative analytic function on $\{z\in(\xi_F,\infty)\}$. In fact, z is an analytic function of $\hat{v}\hat{w}$ on $\{\hat{v}\hat{w}\neq -1\}$ (that last set corresponds to $z=\infty \Leftrightarrow Y=0$, we will return to this shortly):

$$z = \frac{\xi_3+\xi_2\hat{v}\hat{w}}{1+\hat{v}\hat{w}}\,. \qquad (2.9.19)$$

In the $(\hat{w},\hat{v},\hat{\psi},x,\varphi)$ coordinates, one obtains the coefficients of the metric from (2.9.13) using

$$\begin{gathered}
g_{\hat{v}\hat{v}} = \tfrac{1}{c^2\hat{v}^2}g_{vv}\,, \quad g_{\hat{w}\hat{w}} = \tfrac{1}{c^2\hat{w}^2}g_{ww}\,, \\
g_{\hat{v}\hat{w}} = -\tfrac{1}{c^2\hat{v}\hat{w}}g_{vw}\,, \quad g_{\hat{v}\hat{\psi}} = \tfrac{1}{c\hat{v}}g_{v\hat{\psi}}\,, \quad g_{\hat{w}\hat{\psi}} = -\tfrac{1}{c\hat{w}}g_{w\hat{\psi}}\,. \qquad (2.9.20)
\end{gathered}$$

In order to show that the coefficients of the metric are analytic on the set

$$\left\{\hat{w},\hat{v}\mid z(\hat{v}\hat{w})>\xi_F\right\} = \left\{\hat{w},\hat{v}\mid -1<\hat{v}\hat{w}<\frac{\xi_3-\xi_F}{\xi_F-\xi_2}\right\} \qquad (2.9.21)$$

it is convenient to write

$$\begin{gathered}
g_{\hat{v}\hat{v}} = \frac{1}{c^2\hat{v}^2\hat{w}^2}\hat{w}^2 g_{vv}\,, \quad g_{\hat{w}\hat{w}} = \frac{1}{c^2\hat{v}^2\hat{w}^2}\hat{v}^2 g_{ww}\,, \\
g_{\hat{v}\hat{w}} = -\frac{1}{c^2\hat{v}\hat{w}}g_{vw}\,, \quad g_{\hat{v}\hat{\psi}} = \frac{1}{c\hat{v}\hat{w}}\hat{w}g_{v\hat{\psi}}\,, \quad g_{\hat{w}\hat{\psi}} = -\frac{1}{c\hat{v}\hat{w}}\hat{v}g_{w\hat{\psi}}\,. \qquad (2.9.22)
\end{gathered}$$

Hence, to make sure that all the coefficients of metric are well behaved at $\{\hat{w},\hat{v}\in\mathbb{R}\mid z=\xi_3\}$ (i.e. $\hat{v}=0$ or $\hat{w}=0$), it suffices to check that there is a multiplicative factor $(z-\xi_3)^2$ in $g_{vv}=g_{ww}$, as well as a multiplicative factor $(z-\xi_3)$ in g_{vw} and in $g_{v\hat{\psi}}=g_{w\hat{\psi}}$. In view of (2.9.6)–(2.9.9), one can see that this will be the case if , first, a is chosen so that $1+a\sigma(\xi_1-z)=a\sigma(\xi_3-z)$, that is

$$a = \frac{1}{\sigma(\xi_3-\xi_1)}, \qquad (2.9.23)$$

and then, if ξ_0 and b and chosen such that

$$0 = -\frac{a^2\nu\xi_F(\xi_3-\xi_1)(\xi_3-\xi_2)}{\xi_3-\xi_F} + \frac{(\xi_3-\xi_0)^2}{\nu b^2(\xi_3-\xi_1)(\xi_3-\xi_2)}. \qquad (2.9.24)$$

With the choice $\xi_0 = \xi_2$, (2.9.24) will hold if we set

$$b^2 = \frac{(\xi_3-\xi_F)}{\nu^2 a^2 \xi_F(\xi_3-\xi_1)^2}. \qquad (2.9.25)$$

So far we have been focussing on the region $z \in (\xi_F, \infty)$, which overlaps only with part of the manifold "$\{z \in (\xi_3, \infty] \cup [-\infty, \xi_1]\}$". A well behaved coordinate on that last region is $Y = -1/z$. This allows one to go smoothly through $Y = 0$ in (2.9.17):

$$\hat v \hat w = -\frac{1+\xi_3 Y}{1+\xi_2 Y} \quad \Longleftrightarrow \quad Y = -\frac{1+\hat v\hat w}{\xi_3+\xi_2\hat v\hat w}. \qquad (2.9.26)$$

In other words, $\hat v\hat w$ extends analytically to the region of interest, $0 \le Y \le -1/\xi_1$ (and in fact beyond, but this is irrelevant to us). Similarly, the determinant $\det(g_{(\hat w,\hat v,\hat\psi,x,\varphi)})$ extends analytically across $Y = 0$, being the ratio of two polynomials of order eight in z (equivalently, in Y), with limit

$$\det(g_{(\hat w,\hat v,\hat\psi,x,\varphi)}) \to_{z\to\infty} -\frac{F^2(x)}{4A^8 b^2 c^4 \xi_F^4}. \qquad (2.9.27)$$

We conclude that the construction so far produces an analytic Lorentzian metric on the set

$$\hat\Omega := \left\{\hat w, \hat v \mid -\frac{\xi_3-\xi_1}{\xi_2-\xi_1} \le \hat v\hat w < \frac{\xi_3-\xi_F}{\xi_F-\xi_2}\right\} \times S^1_{\hat\psi} \times S^2_{(x,\varphi)}, \qquad (2.9.28)$$

Here a subscript on S^k points to the names of the corresponding local variables.

The map

$$(\hat w, \hat v, \hat\psi, x, \varphi) \mapsto (-\hat w, -\hat v, -\hat\psi, x, -\varphi) \qquad (2.9.29)$$

is an orientation-preserving analytic isometry of the analytically extended metric on $\hat\Omega$. It follows that the manifold

$$\widehat{\mathscr{M}}$$

obtained by gluing together $\hat\Omega$ and two isometric copies of $(\mathscr{M}_I, g)$ can be equipped with the obvious Lorentzian metric, denoted by $\widehat{g}$, which is furthermore analytic. The second copy of $(\mathscr{M}_I, g)$ will be denoted by $(\mathscr{M}_{III}, g)$; compare Figure 2.9.1. The reader should keep in mind the polar character of the coordinates around the relevant axes of rotation, and the special character of the "point at infinity" $z = \xi_1 = x$.

2.10 Global structure

Our discussion of the global structure of $(\widehat{\mathscr{M}}, \widehat{g})$ follows closely [71].

The reader is referred to [71] for a rather involved proof of global hyperbolicity of $\widehat{\mathscr{M}}$.

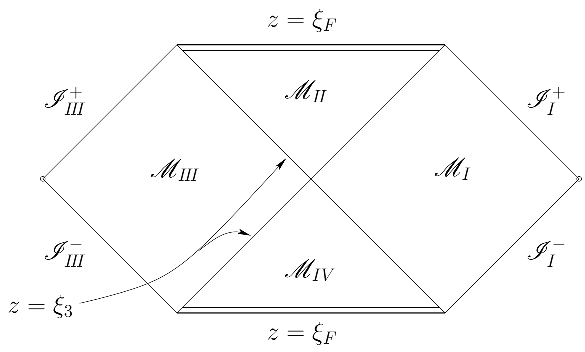

Figure 2.9.1: $\widehat{\mathscr{M}}$ with its various subsets. For example, $\mathscr{M}_{I\cup II}$ is the union of $\mathscr{M}_I$ and of $\mathscr{M}_{II}$ and of that part of $\{z = \xi_3\}$ which lies in the intersection of their closures; this is the manifold constructed in [112]. Very roughly speaking, the various $\mathscr{I}$'s correspond to $x = z = \xi_1$. It should be stressed that this is neither a *conformal* diagram, nor is the spacetime a product of the figure times $S^2 \times S^1$: $\mathscr{M}_I$ cannot be the product of the depicted diamond with $S^2\times S^1$, as this product is not simply connected, while $\mathscr{M}_I$ is. But the diagram represents accurately the causal relations between the various $\mathscr{M}_N$'s, as well as the geometry near the bifurcate horizon $z = \xi_F$, as the manifold *does have* a product structure there.

2.10.1 The event horizon has $S^2 \times S^1 \times \mathbb{R}$ topology

The analysis in Section 2.9 shows that the set

$$\mathscr{E} := \{z = \xi_3\}$$

is a bifurcate Killing horizon. In this section we wish to show that a subset of $\mathscr{E}$ is actually a black-hole event horizon, with $S^2 \times S^1 \times \mathbb{R}$ topology.

Now,

$$g(\nabla z, \nabla z) = g^{zz} = -\frac{A^2(x-z)^2G(z)}{F(x)F(z)} \tag{2.10.1}$$

in the region $\{z > \xi_3\}$, and by analyticity this equation remains valid on $\{z > \xi_F\}$. Equation (2.10.1) shows that $\mathscr{E}$ is a null hypersurface, with z being a time function on $\{\xi_F < z < \xi_3\}$. The usual choice of time orientation implies that z is strictly decreasing along future directed causal curves in the region $\{\hat v > 0\,, \hat w > 0\}$, and strictly increasing along such curves in the region $\{\hat v < 0\,, \hat w < 0\}$. In particular no causal future directed curve can leave the region $\{\hat v > 0\,, \hat w > 0\}$. Hence the spacetime contains a black hole region.

However, it is not clear that $\mathscr{E}$ is the *event horizon* within the Emparan-Reall spacetime $(\mathscr{M}_{I\cup II}, g)$, because the actual event horizon could be enclosing the region $z < \xi_3$. To show that this is not the case, consider the “area function”, defined as the determinant, say W, of the matrix

$$g(K_i, K_j)\,,$$

where the K_i's, $i = 1, 2, 3$, are the Killing vectors equal to ∂_t, ∂_ψ, and ∂_φ in the asymptotically flat region. In the coordinates of (2.5.4) this equals

$$\frac{F(x)G(x)F(z)G(z)}{A^4(x-z)^4}\,. \tag{2.10.2}$$

Analyticity implies that this formula is valid throughout $\mathscr{M}_{I\cup II}$, as well as $\widehat{\mathscr{M}}$. Now,

$$F(z)G(z)=\frac{\nu}{\xi_F}(\xi_F-z)(z-\xi_1)(z-\xi_2)(z-\xi_3)\,,$$

and, in view of the range of the variable x, the sign of (2.10.2) depends only upon the values of z. Since $F(z)G(z)$ behaves as $-\nu z^4/\xi_F$ for large z, W is negative both for $z<\xi_1$ and for $z>\xi_3$. Hence, at each point p of those two regions the set of vectors in $T_p\mathscr{M}$ spanned by the Killing vectors is timelike. So, suppose for contradiction, that the event horizon $\mathscr{H}$ intersects the region $\{z\in(\xi_3,\infty]\}\cup\{z\in[-\infty,\xi_1)\}$; here "$z=\pm\infty$" should be understood as $Y=0$, as already mentioned in the introduction. Since $\mathscr{H}$ is a null hypersurface invariant under isometries, every Killing vector is tangent to $\mathscr{H}$. However, at each point at which W is negative there exists a linear combination of the Killing vectors which is timelike. This gives a contradiction because no timelike vectors are tangent to a null hypersurface.

We conclude that $\{z=\xi_3\}$ forms indeed the event horizon in the spacetime $(\mathscr{M}_{I\cup II},g)$, with topology $\mathbb{R}\times S^1\times S^2$.

The argument just given also shows that the domain of outer communications within $(\mathscr{M}_I,g)$ coincides with $(\mathscr{M}_I,g)$.

Similarly, one finds that the domain of outer communications within $(\widehat{\mathscr{M}},\widehat{g})$, or that within $(\mathscr{M}_{I\cup II},g)$, associated with an asymptotic region lying in $(\mathscr{M}_I,g)$, is $(\mathscr{M}_I,g)$. The boundary of the d.o.c. in $(\widehat{\mathscr{M}},\widehat{g})$ is a subset of the set $\{z=\xi_3\}$, which can be found by inspection of Figure 2.9.1.

2.10.2 Inextendibility at $z=\xi_F$, maximality

The obvious place where $(\widehat{\mathscr{M}},\widehat{g})$ could be enlarged is at $z=\xi_F$. To show that no extension is possible there, note that the Lorentzian length of the Killing vector ∂_t satisfies

$$g(\partial_t,\partial_t)=-\frac{F(x)}{F(z)}\to_{\xi_F<z\to\xi_F}\infty\quad\text{(recall that } F(x)\geq 1-\tfrac{\xi_2}{\xi_F}>0). \tag{2.10.3}$$

Inextendibility of the spacetime across the boundary $\{z=\xi_F\}$ follows from this and from Theorem 1.4.2, p. 57.

An alternative way, demanding somewhat more work, of proving that the Emparan-Reall metric is C^2–inextendible across $\{z=\xi_F\}$, is to notice that $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded along any curve along which z approaches ξ_F. This has been pointed out to us by Harvey Reall (private communication), and has been further verified by Alfonso Garcia-Parrado and José María Martín García using the symbolic algebra package xAct [199]:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=\frac{12A^4\xi_F^4G(\xi_F)^2(x-z)^4\,(1+O(z-\xi_F))}{(\xi_F-x)^2(z-\xi_F)^6}\,. \tag{2.10.4}$$

(Note that the factor $(x-z)^4$ is strictly bounded away from zero for $z \to \xi_F$.)

The following result is established in [71]:

THEOREM 2.10.1 *All maximally extended causal geodesics in* $(\widehat{\mathscr{M}}, \widehat{g})$ *are either complete, or reach a singular boundary* $\{z = \xi_F\}$ *in finite affine time.*

This, together with Proposition 1.4.3 gives:

THEOREM 2.10.2 $(\widehat{\mathscr{M}}, \widehat{g})$ *is maximal within the class of* C^2 *Lorentzian manifolds.*

2.10.3 Conformal infinity $\mathscr{I}$

In this section we address the question of existence of conformal completions at null infinity *à la Penrose*, for a class of higher dimensional stationary spacetimes that includes the Emparan-Reall metrics; see the Appendix of [104] and [102] for the $3+1$ dimensional case.

We start by noting that any stationary asymptotically flat spacetime which is vacuum, or electro-vacuum, outside of a spatially compact set is necessarily *asymptotically Schwarzschildian*, in the sense that there exists a coordinate system in which the leading order terms of the metric have the Schwarzschild form, with the error terms falling-off one power of r faster:

$$g = g_m + O(r^{-(n-1)}) \tag{2.10.5}$$

in spacetime dimension $n+1$, where g_m is the Schwarzschild metric of mass m, and the size of the decay of the error terms in (2.10.5) is measured in a manifestly asymptotically Minkowskian coordinate system. The proof of this fact is outlined briefly in [20, Section 2]. In that last reference it is also shown that the remainder term has a full asymptotic expansion in terms of inverse powers of r in dimension $2k+1$, $k \geq 3$, or in dimension $4+1$ for static metrics. Otherwise, the remainder is known to have an asymptotic expansion in terms of inverse powers of r and of $\ln r$, and whether or not there will be non-trivial logarithmic terms in the expansion is not known in general.

In higher dimensions, the question of existence of a conformal completion at null infinity is straightforward: We start by writing the $(n+1)$–dimensional Minkowski metric as

$$\eta = -dt^2 + dr^2 + r^2 h\,, \tag{2.10.6}$$

where h is the round unit metric on an $(n-2)$-dimensional sphere. Replacing t by the standard retarded time $u = t - r$, one is led to the following form of the metric g:

$$g = -du^2 - 2du\,dr + r^2 h + O(r^{-(n-2)})dx^\mu dx^\nu\,, \tag{2.10.7}$$

where the dx^μ's are the manifestly Minkowskian coordinates $(t, x^1, \ldots, x^n)$ coordinates for η. Setting $x = 1/r$ in (2.10.7) one obtains

$$g = \frac{1}{x^2}\Big(-x^2du^2 + 2du\,dx + h + O(x^{n-4})dy^\alpha dy^\beta\Big)\,, \tag{2.10.8}$$

with correction terms in (2.10.8) which will extend smoothly to $x = 0$ in the coordinate system $(y^\mu) = (u, x, v^A)$, where the v^A's are local coordinates on S^{n-2}. For example, a term $O(r^{-2})dx^i dx^j$ in g will contribute a term

$$O(r^{-2})dr^2 = O(r^{-2})x^{-4}dx^2 = x^{-2}(O(1)dx^2)\,,$$

which is bounded up to $x = 0$ after a rescaling by x^2. The remaining terms in (2.10.8) are analysed similarly.

In dimension $4 + 1$, care has to be taken to make sure that the correction terms do not affect the signature of the metric so extended; in higher dimension this is already apparent from (2.10.8).

So, to construct a conformal completion at null infinity for the Emparan Reall metric it suffices to verify that the determinant of the conformally rescaled metric, when expressed in the coordinates described above, does not vanish at $x = 0$. This is indeed the case, and can be seen by calculating the Jacobian of the map

$$(t, z, \psi, x, \varphi) \mapsto (u, x, v^A)\,;$$

the result can then be used to calculate the determinant of the metric in the new coordinates, making use of the formula for the determinant of the metric in the original coordinates.

For a general stationary vacuum $4 + 1$ dimensional metric one can always transform to the coordinates, alluded to above, in which the metric is manifestly Schwarzschildian in leading order. Instead of using $(u = t - r, x = 1/r)$ one can use coordinates $(u_m, x = 1/r)$, where u_m is the corresponding null coordinate u for the $4+1$ dimensional Schwarzschild metric. This will lead to a conformally rescaled metric with the correct signature on the conformal boundary. Note, however, that this transformation might introduce log terms in the metric, even if there were none to start with; this is why we did not use this above.

In summary, whenever a stationary, vacuum, asymptotically flat, $(n + 1)$–dimensional metric, $4 \neq n \geq 3$, has an asymptotic expansion in terms of inverse powers of r, one is led to a smooth $\mathscr{I}$. This is the case for any such metric in dimensions $3 + 1$ or $2k + 1$, $k \geq 3$. In the remaining dimensions one always has a polyhomogeneous conformal completion at null infinity, with a conformally rescaled metric which is C^{n-4} up-to-boundary. For the Emparan-Reall metric there exists a completion which has no logarithmic terms, and is thus C^∞ up-to-boundary.

2.10.4 Uniqueness and non-uniqueness of extensions

In Section 1.4.1 we have seen several examples of non-uniqueness of maximal analytic extensions of an analytic spacetime, which all apply here.

It turns out that a further example of non-uniqueness, analogous to the $\mathbb{RP}^3$ geon of Example 1.4.1, can be constructed for the black ring solution. Indeed, let $(\widehat{\mathscr{M}}, \widehat{g})$ be our extension, as constructed above, of the domain of outer communication $(\mathscr{M}_I, g)$ within the Emparan-Reall spacetime $(\mathscr{M}_{I\cup II}, g)$, and let $\Psi : \widehat{\mathscr{M}} \to \widehat{\mathscr{M}}$ be defined as

$$\Psi(\hat{v}, \hat{w}, \hat{\psi}, x, \varphi) = (\hat{w}, \hat{v}, \hat{\psi} + \pi, x, -\varphi)\,. \tag{2.10.9}$$

By inspection of (2.9.6)-(2.9.8) and (2.9.20), the map Ψ is an isometry, and clearly satisfies the three conditions a), b) and c) spelled out on p. 56. Then $\widehat{\mathscr{M}}/\Psi$ is a maximal, orientable, time-orientable, analytic extension of $\mathscr{M}_I$ distinct from $\widehat{\mathscr{M}}$.

Further similar examples can be constructed using isometries which do not preserve orientation, and/or time-orientation.

On the positive side, we have the following uniqueness result for our extension $(\widehat{\mathscr{M}}, \widehat{g})$ of the Emparan-Reall spacetime $(\mathscr{M}_I, g)$, which follows immediately from Corollary 1.4.7 and from the properties of causal geodesics of $(\widehat{\mathscr{M}}, \widehat{g})$ spelled-out in Theorem 2.10.1:

THEOREM 2.10.3 *$(\widehat{\mathscr{M}}, \widehat{g})$ is unique within the class of simply connected analytic extensions of $(\mathscr{M}_I, g)$ which have the property that all maximally extended causal geodesics on which $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded are complete.*

As usual, uniqueness here is understood up to isometry.

The examples presented in Section 1.4.1 show that the hypotheses of Theorem 2.10.3 are optimal.

It is natural to raise another uniqueness question, namely of the singling-out features of the Emparan-Reall metric amongst all five-dimensional vacuum stationary black-hole spacetimes? Partial answers to this can be found in the work of Hollands and Yazadjiev [153].

2.10.5 Other coordinate systems

An alternative convenient form of the Emparan-Reall metric has been given in [113]:

$$\begin{aligned} g &= \frac{R^2F(x)}{(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{G(x)}{F(x)}d\varphi^2 - \frac{G(y)}{F(y)}d\psi^2\right) \\ &\quad -\frac{F(y)}{F(x)}\left(dt - \frac{CR(1+y)}{F(y)}d\psi\right)^2 , \end{aligned} \tag{2.10.10}$$

where

$$F(z) = 1 + \lambda z, \quad G(z) = (1-z^2)(1+\nu z), \quad C = \sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}} ,$$

with

$$\lambda = \frac{2\nu}{1+\nu^2} , \qquad 0 < \nu < 1 .$$

If we denote by $\{\hat{t}, \hat{x}, \hat{y}, \hat{\psi}, \hat{\varphi}\}$ the original coordinates of (2.0.1), then we have the relation

$$t = \hat{t}, \quad x = \frac{\hat{\lambda} - \hat{x}}{-1 + \hat{\lambda}\hat{x}}, \quad y = \frac{\hat{\lambda} - \hat{y}}{-1 + \hat{\lambda}\hat{y}}, \quad \varphi = \frac{1 - \hat{\lambda}\hat{\nu}}{\sqrt{1 - \hat{\lambda}^2}}\hat{\varphi}, \quad \psi = \frac{1 - \hat{\lambda}\hat{\nu}}{\sqrt{1 - \hat{\lambda}^2}}\hat{\psi}, \tag{2.10.11}$$

where

$$\hat{\nu} = \frac{\nu - \lambda}{\lambda\nu - 1}, \quad \hat{\lambda} = \lambda, \quad \nu = \frac{\hat{\nu} - \hat{\lambda}}{\hat{\lambda}\hat{\nu} - 1}.$$

The transformation (2.10.11) brings the metric (2.10.10) into the form

$$g = -\frac{\hat{F}(\hat{x})}{\hat{F}(\hat{y})}(d\hat{t} + \hat{A}\sqrt{\hat{\lambda}\hat{\nu}}(1+\hat{y})d\hat{\psi})^2 + \frac{\hat{A}^2}{(\hat{x}-\hat{y})^2}\times$$
$$\left[\hat{F}(\hat{y})^2\left(\frac{d\hat{x}^2}{\hat{G}(\hat{x})} + \frac{\hat{G}(\hat{x})}{\hat{F}(\hat{x})}d\hat{\varphi}^2\right) - \hat{F}(\hat{x})\left(\frac{\hat{F}(\hat{y})}{\hat{G}(\hat{y})}d\hat{y}^2 + \hat{G}(\hat{y})d\hat{\psi}^2\right)\right], \quad (2.10.12)$$

where

$$\hat{F}(z) = 1 - \hat{\lambda}z, \quad \hat{G}(z) = (1-z^2)(1-\hat{\nu}z), \quad \hat{A} = -R\sqrt{\frac{(1-\hat{\lambda}\hat{\nu})}{1-\hat{\lambda}^2}}\,.$$

Simple rescalings and redefinitions of constants bring (2.10.12) to the form (2.0.1).

Chapter 3

Rasheed's Kaluza-Klein black holes

Kaluza-Klein solutions have attracted a lot of attention as interesting models in theoretical physics. In vacuum, and with vanishing cosmological constant, the simplest such solutions are obtained by taking the product of a known vacuum solution with a flat torus. A more sophisticated class of black-hole solutions of this kind has been discovered by Rasheed in [240]. The aim of this short chapter is to discuss their geometry.

3.1 Rasheed's metrics

In [240] D. Rasheed constructed a family of stationary axi-symmetric solutions of the five-dimensional vacuum Einstein equations. The globlal structure of those solutions has been analysed in [15], we follow the presentation there.

The Rasheed metrics take the form

$$ds^2_{(5)} = \frac{B}{A}\left(dx^4 + 2A_\mu dx^\mu\right)^2 + \sqrt{\frac{A}{B}}ds^2_{(4)}\,, \tag{3.1.1}$$

where a, M, P, Q and Σ are real numbers satisfying

$$\tfrac{Q^2}{\Sigma+M\sqrt{3}} + \tfrac{P^2}{\Sigma-M\sqrt{3}} = \tfrac{2\Sigma}{3}\,, \tag{3.1.2}$$

$$M^2+\Sigma^2-P^2-Q^2 \neq 0\,, \quad \left(M+\Sigma/\sqrt{3}\right)^2 - Q^2 \neq 0\,, \quad \left(M-\Sigma/\sqrt{3}\right)^2 - P^2 \neq 0\,, \tag{3.1.3}$$

$$M \pm \tfrac{\Sigma}{\sqrt{3}} \neq 0\,, \quad F^2 := \frac{\left[\left(M+\Sigma/\sqrt{3}\right)^2-Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2-P^2\right]}{M^2+\Sigma^2-P^2-Q^2} > 0\,, \tag{3.1.4}$$

and where

$$ds^2_{(4)} = -\frac{G}{\sqrt{AB}}\left(dt+\omega^0{}_\phi d\phi\right)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\Delta\sqrt{AB}}{G}\sin^2(\theta)d\phi^2\,, \tag{3.1.5}$$

with

$$A = \left(r-\Sigma/\sqrt{3}\right)^2 - \frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} + a^2\cos^2(\theta) + \frac{2JPQ\cos(\theta)}{\left(M+\Sigma/\sqrt{3}\right)^2-Q^2}\,,$$

$$
\begin{aligned}
B &= \left(r + \Sigma/\sqrt{3}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2\cos^2(\theta) - \frac{2JPQ\cos(\theta)}{\left(M - \Sigma/\sqrt{3}\right)^2 - P^2}\,, \\
G &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2(\theta)\,, \\
\Delta &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\,, \\
\omega^0{}_\phi &= \frac{2J\sin^2(\theta)}{G}\left[r + E\right]\,, \\
J^2 &= a^2F^2\,, \qquad (3.1.6)
\end{aligned}
$$

whereas E is given by

$$
E = -M + \frac{\left(M^2 + \Sigma^2 - P^2 - Q^2\right)\left(M + \Sigma/\sqrt{3}\right)}{\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2}\,. \qquad (3.1.7)
$$

In Kaluza-Klein theories, vacuum metrics on the Kaluza-Klein bundle lead to solutions of the Einstein-Maxwell-dilaton field equations. In the Rasheed case the physical-space Maxwell potential is given by

$$
2A_\mu dx^\mu = \frac{C}{B}dt + \left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)d\phi\,, \qquad (3.1.8)
$$

where

$$
\begin{aligned}
C &= 2Q\left(r - \Sigma/\sqrt{3}\right) - \frac{2PJ\cos(\theta)\left(M + \Sigma/\sqrt{3}\right)}{\left(M - \Sigma/\sqrt{3}\right)^2 - P^2}\,, \qquad (3.1.9) \\
\omega^5{}_\phi &= \frac{H}{G}\,, \qquad (3.1.10)
\end{aligned}
$$

and

$$
H := 2P\Delta\cos(\theta) - \frac{2QJ\sin^2(\theta)\left[r\left(M - \Sigma/\sqrt{3}\right) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2\right]}{\left[\left(M + \Sigma/\sqrt{3}\right)^2 - Q^2\right]}\,. \qquad (3.1.11)
$$

The Rasheed metrics (3.1.1) have been obtained by applying a solution-generating technique ([240], compare [131]) to the Kerr metrics. This guarantees that these metrics solve the five-dimensional vacuum Einstein equations when the constraint (3.1.3) is satisfied.

Let us address the question of the global structure of the metrics above. We have

$$
\det g = -A^2\sin^2(\theta)\,,
$$

which shows that the metrics are smooth and Lorentzian except possibly at the zeros of A, B, G, Δ, and $\sin(\theta)$.

After a suitable periodicity of ϕ as in Section 3.4 below has been imposed, regularity at the axes of rotation away from the zeros of denominators follows from the factorisations

$$\left(\frac{\Delta}{G}-1\right) = \frac{a^2\sin^2(\theta)}{a^2\cos^2(\theta)-2Mr+P^2+Q^2+r^2-\Sigma^2}, \tag{3.1.12}$$

$$2A_\phi - 2P\frac{\Delta}{G}\cos(\theta) = \frac{\sin^2(\theta)}{G}\left(\mathcal{H}+\frac{2JC}{B}\left[r+E\right]\right), \tag{3.1.13}$$

where

$$\mathcal{H} := -\frac{2QJ\left[r\left(M-\Sigma/\sqrt{3}\right)+M\Sigma/\sqrt{3}+\Sigma^2-P^2-Q^2\right]}{\left[\left(M+\Sigma/\sqrt{3}\right)^2-Q^2\right]}. \tag{3.1.14}$$

It will be seen below that, after restricting the parameter ranges as in (3.2.4) and (3.2.6), the location of Killing horizons is determined by the zeros of

$$\begin{vmatrix} g_{tt} & g_{t\phi} & g_{t4} \\ g_{\phi t} & g_{\phi\phi} & g_{\phi 4} \\ g_{4t} & g_{4\phi} & g_{44} \end{vmatrix} = -\Delta\sin^2(\theta), \tag{3.1.15}$$

and thus by the real roots $r_+ \geq r_-$ of Δ, if any:

$$r_\pm = M \pm \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2}. \tag{3.1.16}$$

3.2 Zeros of the denominators

The norms

$$g_{tt} = \frac{W}{AB} \text{ and } g_{44} = \frac{B}{A},$$

of the Killing vectors ∂_t and ∂_4 are geometric invariants, where $W = -GA+C^2$. So zeros of A and of AB correspond to singularities in the five-dimensional geometry except if

1. a zero of A is a joint zero of A, B and W, or if
2. a zero of B which is not a zero of A is also a zero of W.

Setting

$$\mathcal{A} := \frac{2JPQ}{a^2\left(\left(M+\Sigma/\sqrt{3}\right)^2-Q^2\right)}, \tag{3.2.1}$$

one checks that if

$$\begin{cases} \frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} - a^2(1-|\mathcal{A}|) = 0, & \text{when } |\mathcal{A}|>2 \quad \text{or} \\ \frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} = 0, & \text{when } |\mathcal{A}|\leq 2, \end{cases} \tag{3.2.2}$$

then A vanishes exactly at one point. Otherwise the set of zeros of A forms a curve in the (r,θ) plane. Let $\theta \mapsto r_A^+(\theta)$ denote the curve, say γ, corresponding to the set of largest zeros of A.

Note that W and A are polynomials in r, with A of second order. If W/A is smooth, the remainder of the polynomial division of W by $r-r_A^+$ must vanish on the part of γ that lies outside the horizon. One can calculate this remainder with MATHEMATICA, obtaining a function of θ which vanishes at most at isolated points, if at all. It follows that the division of W by A is singular on the closure of the domain of outer communications (d.o.c.), i.e. the region $\{r \geq r_+\}$, if A has zeros there, except perhaps when (3.2.2) holds.

One can likewise exclude a joint zero of W and B in the closure of the d.o.c. without a zero of A, except possibly for the case where this zero is isolated for B as well, which happens if

$$\begin{cases} \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathcal{B}|) = 0, & \text{if } |\mathcal{B}| > 2 \quad \text{or} \\ \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} = 0, & \text{if } |\mathcal{B}| \leq 2\,. \end{cases} \tag{3.2.3}$$

See [157] for a more detailed analysis of the borderline cases.

Summarising: a necessary condition for a black hole without obvious singularities in the closure of the domain of outer communications is that all zeros of A lie under the outermost Killing horizon $r = r_+$. One finds that this will be the case if and only if

$$|\mathcal{A}| > 2 \quad \text{and} \quad \begin{cases} \frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} - a^2(1-|\mathcal{A}|) < 0, & \text{or} \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} - a^2(1-|\mathcal{A}|)}, \end{cases}$$

or

$$|\mathcal{A}| \leq 2 \quad \text{and} \quad \begin{cases} \frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} < 0, & \text{or} \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma-M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4}}, \end{cases} \tag{3.2.4}$$

except perhaps when (3.2.2) holds.

An identical argument applies to the zeros of B, with the zeros of B lying on a curve unless (3.2.3) holds. Ignoring this last case, the zeros of B need similarly be hidden behind the outermost Killing horizon. Setting

$$\mathcal{B} := -\frac{2JPQ}{a^2\left(\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right)}\,, \tag{3.2.5}$$

one finds that this will be the case if and only if

$$|\mathcal{B}| > 2 \quad \text{and} \quad \begin{cases} \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathcal{B}|) < 0, & \text{or} \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} - a^2(1-|\mathcal{B}|)}, \end{cases}$$

or

$$|\mathcal{B}| \leq 2 \quad \text{and} \quad \begin{cases} \frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} < 0, & \text{or} \\ M + \sqrt{M^2+\Sigma^2-P^2-Q^2-a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma+M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4}}, \end{cases} \tag{3.2.6}$$

except perhaps when (3.2.3) holds.

While the above guarantees lack of obvious singularities in the *domain of outer communications* $\{r > r_+\}$ (d.o.c.), there could still be causality violations there. Ideally the d.o.c. should be globally hyperbolic, a question which we have not attempted to address. Barring global hyperbolicity, a decent d.o.c. should at least admit a time function, and the function t provides an obvious candidate. In order to study the issue we note the identity

$$g^{00} \quad = \quad \frac{4J^2[r+E]^2\sin^2(\theta) - AB\Delta}{A\Delta G}\,. \tag{3.2.7}$$

A Mathematica calculation shows that the numerator factorises through G, so that g^{00} extends smoothly through the ergosphere. When $P = 0$ one can verify that g^{00} is negative on the d.o.c. For $P \neq 0$ one can find open sets of parameters which guarantee that g^{00} is strictly negative for $r > r_+$ when A and B have no zeros there. An example is given by the condition

$$r_+ \geq \frac{EM+q}{M+E}\,, \tag{3.2.8}$$

which is sufficient but not necessary, where $q := P^2 + Q^2 - \Sigma^2 + a^2$. We hope to return to the question of causality violations in the future.

In Figure 3.2.1 we show the locations of the zeros of A and B for some specific sets of parameters satisfying, or violating, the conditions above.

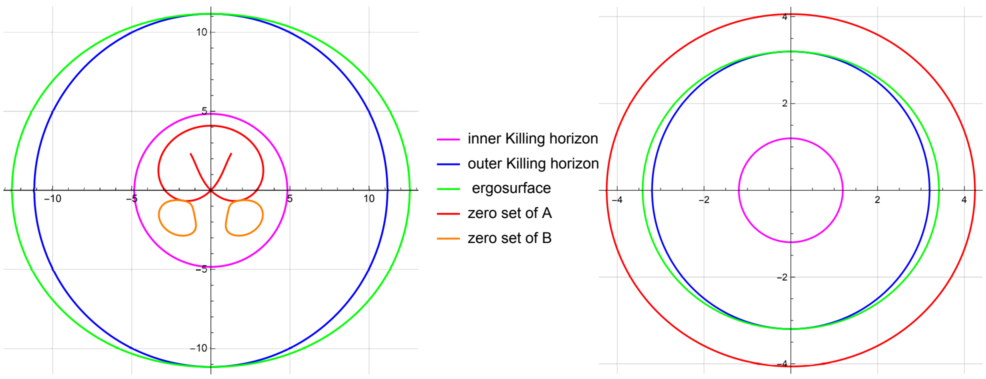

Figure 3.2.1: Two sample plots for the location of the ergosurface (zeros of G), the outer and inner Killing horizons (zeros of Δ), and the zeros of $A\,,B$. Left plot: $M = 8$, $a = \frac{33}{10}$, $Q = \frac{8}{5}$, $\Sigma = -\frac{23}{5}$, $P = -\frac{1}{5}\sqrt{\frac{2\left(4105960\sqrt{3}+2770943\right)}{12813}} \approx -7.86$, with zeros of A and B under both horizons, consistently with (3.2.4) and (3.2.6). Right plot: $M = 1$, $a = 1$, $Q = 0$, $\Sigma = \sqrt{6}$, $P = \sqrt{4-2\sqrt{2}} \approx 1.08$; here (3.2.4) is violated, while the zeros of B occur at negative r.

Another potential source of singularities of the metric (3.1.1) could be the zeros of G. It turns out that there are irrelevant, which can be seen as follows: The relevant metric coefficient is $g_{\phi\phi}$, which reads

$$g_{\phi\phi} = \frac{B}{A}\left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)^2 + \sqrt{\frac{A}{B}}\left(-\frac{G}{\sqrt{AB}}\left(\omega^0{}_\phi\right)^2 + \frac{\Delta\sqrt{AB}}{G}\sin^2\theta\right). \tag{3.2.9}$$

Taking into account a G^{-1} factor in $\omega^0{}_\phi$, it follows that $g_{\phi\phi}$ can be written as a fraction $(\ldots)/ABG^2$. A Mathematica calculation shows that the denominator $(\ldots)$ factorises through AG^2, which shows indeed that the zeros of G are innocuous for the problem at hand.

Let us write $ds^2_{(4)}$ as ${}^{(4)}g_{ab}dx^a dx^b$. The factorisation just described works for $g_{\phi\phi}$ but does *not* work for ${}^{(4)}g_{\phi\phi}$. From what has been said we see that the quotient metric ${}^{(4)}g_{ab}dx^a dx^b$ is always singular in the d.o.c., a fact which seems to have been ignored, and unnoticed, in the literature so far.

3.3 Regularity at the outer Killing horizon $\mathcal{H}_+$

The location of the outer Killing horizon $\mathcal{H}_+$ of the Killing field

$$k = \partial_t + \Omega_\phi \partial_\phi + \Omega_4 \partial_{x^4}\,, \tag{3.3.1}$$

is given by the larger root r_+ of Δ, cf. (3.1.16). The condition that $\mathcal{H}_+$ is a Killing horizon for k is that the pullback of $g_{\mu\nu}k^\nu$ to $\mathcal{H}_+$ vanishes. This, together with

$$\Delta|_{\mathcal{H}_+} = 0\,, \quad G|_{\mathcal{H}_+} = -a^2 \sin^2(\theta)\,, \tag{3.3.2}$$

yields

$$\begin{aligned}
\Omega_\phi &= -\frac{1}{\omega^0{}_\phi}\Big|_{\mathcal{H}_+} = \frac{a^2}{2J}(r_+ + E)^{-1}\,,\\
\Omega_4 &= -\frac{2(A_t\omega^0{}_\phi - A_\phi)}{\omega^0{}_\phi}\Big|_{\mathcal{H}_+}\\
&= \frac{Q\left(-3Mr_+ - \sqrt{3}M\Sigma + 3P^2 + 3Q^2 + \sqrt{3}r\Sigma - 3\Sigma^2\right)}{(E + r_+)\left(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + \Sigma^2\right)}\,.
\end{aligned} \tag{3.3.3}$$

After the coordinate transformation

$$\bar{\phi} = \phi - \Omega_\phi\, dt\,, \quad \bar{x}^4 = x^4 - \Omega_4\, dt\,, \tag{3.3.4}$$

the metric (3.1.1) becomes

$$g = g_S + \frac{dr^2}{\Delta} + \Delta U dt^2\,, \tag{3.3.5}$$

where g_S is a smooth $(0,2)$-tensor, with $U := g_{tt}/\Delta$ extending smoothly across $\Delta = 0$. Introducing a new time coordinate by

$$\tau = t - \sigma \ln(r - r_+) \Rightarrow \quad d\tau = dt - \frac{\sigma}{r - r_+}\, dr\,, \tag{3.3.6}$$

where σ is a constant to be determined, (3.3.5) takes the form

$$\begin{aligned}
g &= g_S + \Delta U \left(d\tau + \frac{\sigma}{r - r_+}dr\right)^2 + \frac{dr^2}{\Delta}\\
&= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \left(\frac{1}{\Delta} + \frac{\Delta U \sigma^2}{(r - r_+)^2}\right) dr^2\\
&= g_S + \Delta U d\tau^2 + \frac{2\Delta U\sigma}{r - r_+}d\tau dr + \underbrace{\frac{(r - r_+)^2 + \Delta^2\sigma^2 U}{\Delta(r - r_+)^2}}_{V} dr^2\,.
\end{aligned} \tag{3.3.7}$$

In order to obtain a smooth metric in the domain of outer communication the constant σ has to be chosen so that the numerator of V has a triple-zero at $r = r_+$. A MATHEMATICA computation gives an explicit formula for the desired constant σ, which is too lengthy to be explicitly presented here. This establishes smooth extendibility of the metric in suitable coordinates across $r = r_+$.

3.4 Asymptotic behaviour

When $P = 0$ the Rasheed metrics satisfy the KK-asymptotic flatness conditions. This can be seen by introducing manifestly-asymptotically-flat coordinates (t, x, y, z) in the usual way. With some work one finds that the metric takes the form

$$\begin{pmatrix} \frac{2M}{r} + \frac{2\Sigma}{\sqrt{3}r} - 1 & 0 & 0 & 0 & \frac{2Q}{r} \\ 0 & \frac{2Mx^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} & 0 \\ 0 & \frac{2Mxy}{r^3} & \frac{2My^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Myz}{r^3} & 0 \\ 0 & \frac{2Mxz}{r^3} & \frac{2Myz}{r^3} & \frac{2Mz^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & 0 \\ \frac{2Q}{r} & 0 & 0 & 0 & \frac{4\Sigma}{\sqrt{3}r} + 1 \end{pmatrix} + O(r^{-2}) \,. \tag{3.4.1}$$

However, when $P \neq 0$, the Rasheed metrics do *not* satisfy the KK-asymptotic flatness requirements anymore: Instead, the space of Rasheed metrics decomposes into sectors, labelled by $P \in \mathbb{R}$, in which the metrics g asymptote to the background metric

$$b := \big(dx^4 + 2P\cos(\theta)d\varphi\big)^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \,. \tag{3.4.2}$$

The metrics (3.1.1) and (3.4.2) are singular at $\sin(\theta) = 0$. This can be resolved by replacing x^4 by $\overline{x}^4$, respectively by $\tilde{x}^4$, on the following coordinate patches:

$$\begin{cases} \overline{x}^4 := x^4 + 2P\varphi \,, & \theta \in [0, \pi) \,, \\ \tilde{x}^4 := x^4 - 2P\varphi \,, & \theta \in (0, \pi] \,. \end{cases} \tag{3.4.3}$$

Indeed, the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\overline{x}^4 + 2P(\cos(\theta) - 1)d\varphi = d\overline{x}^4 - \frac{2P}{r(r+z)}(xdy - ydx)$$

is smooth for $r > 0$ on $\{\theta \in [0, \pi)\}$. Similarly the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\tilde{x}^4 + 2P(\cos(\theta) + 1)d\varphi = d\tilde{x}^4 + \frac{2P}{r(r-z)}(xdy - ydx)$$

is smooth on $\{\theta \in (0, \pi] \,,\, r > 0\}$. Smoothness of both g and b outside of the event horizons readily follows.

We note the relation

$$\overline{x}^4 = \tilde{x}^4 + 4P\varphi \,, \tag{3.4.4}$$

which implies a smooth geometry with periodic coordinates $\overline{x}^4$ and $\tilde{x}^4$ if and only if

$$\text{both } \overline{x}^4 \text{ and } \tilde{x}^4 \text{ are periodic with period } 8P\pi. \tag{3.4.5}$$

From this perspective x^4 is *not a coordinate* anymore: instead the basic coordinates are $\overline{x}^4$ for $\theta \in [0,\pi)$ and $\tilde{x}^4$ for $\theta \in (0,\pi]$, with dx^4 (but *not* x^4) well defined away from the axes of rotation $\{\sin(\theta) = 0\}$ as

$$dx^4 = \begin{cases} d\overline{x}^4 - 2P d\varphi\,, & \theta \in [0,\pi)\,, \\ d\tilde{x}^4 + 2P d\varphi\,, & \theta \in (0,\pi]\,. \end{cases} \tag{3.4.6}$$

Curvature of the asymptotic background

We continue with a calculation of the curvature tensor of the asymptotic background. It is convenient to work in the coframe

$$\overline{\Theta}^{\hat{0}} = dt\,, \qquad \overline{\Theta}^{\hat{1}} = dx\,, \qquad \overline{\Theta}^{\hat{2}} = dy\,, \qquad \overline{\Theta}^{\hat{3}} = dz\,, \qquad \overline{\Theta}^{\hat{4}} = dx^4 + 2P\cos(\theta) d\varphi\,, \tag{3.4.7}$$

which is manifestly smooth after replacing dx^4 as in (3.4.6). Using

$$d\overline{\Theta}^{\hat{4}} = -2P\sin(\theta)\,d\theta\wedge d\varphi = -2P\frac{x^i}{r^3}\partial_i\rfloor(dx\wedge dy\wedge dz) = -\frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^{\hat{i}}dx^{\hat{j}}\wedge dx^{\hat{k}}\,, \tag{3.4.8}$$

where $\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}} \in \{0,\pm 1\}$ denotes the usual epsilon symbol, one finds the following non-vanishing connection coefficients

$$\overline{\omega}^{\hat{4}}{}_{\hat{i}} = \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^{\hat{j}}\overline{\Theta}^{\hat{k}}\,, \quad \overline{\omega}^{\hat{i}}{}_{\hat{j}} = \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^{\hat{k}}\overline{\Theta}^{\hat{4}}\,, \tag{3.4.9}$$

where $x^{\hat{i}} \equiv x^i$. This leads to the following curvature forms

$$\begin{aligned} \overline{\Omega}^{\hat{i}}{}_{\hat{j}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{k}}x^{\hat{\ell}} + \delta^{\hat{k}}_{\hat{\ell}}\right)\overline{\Theta}^{\hat{\ell}}\wedge\overline{\Theta}^{\hat{4}} - \frac{2P^2}{r^6}\mathring{\epsilon}_{\hat{i}\hat{m}(\hat{k}}\epsilon_{\hat{j})\hat{n}\hat{\ell}}x^{\hat{m}}x^{\hat{n}}\overline{\Theta}^{\hat{k}}\wedge\overline{\Theta}^{\hat{\ell}}\,, \\ \overline{\Omega}^{\hat{4}}{}_{\hat{i}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{j}}x^{\hat{\ell}} + \delta^{\hat{j}}_{\hat{\ell}}\right)\overline{\Theta}^{\hat{\ell}}\wedge\overline{\Theta}^{\hat{k}} + \frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{j}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\overline{\Theta}^{\hat{j}}\wedge\overline{\Theta}^{\hat{4}}, \end{aligned} \tag{3.4.10}$$

hence the following non-vanishing curvature tensor components

$$\begin{aligned} \overline{\mathbf{R}}^{\hat{i}}{}_{\hat{j}\hat{k}\hat{4}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{\ell}}\left(-\frac{3}{r^2}x^{\hat{\ell}}x^{\hat{k}} + \delta^{\hat{\ell}}_{\hat{k}}\right)\,, \\ \overline{\mathbf{R}}^{\hat{4}}{}_{\hat{i}\hat{j}\hat{4}} &= \frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{j}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\,, \quad \overline{\mathbf{R}}_{\hat{i}\hat{j}\hat{k}\hat{\ell}} = -\frac{2P^2}{r^6}(\mathring{\epsilon}_{\hat{i}\hat{j}\hat{n}}\mathring{\epsilon}_{\hat{k}\hat{\ell}\hat{m}} + \mathring{\epsilon}_{\hat{i}\hat{m}[\hat{k}}\mathring{\epsilon}_{\hat{\ell}]\hat{j}\hat{n}})x^{\hat{m}}x^{\hat{n}}. \end{aligned} \tag{3.4.11}$$

The non-vanishing components of the Ricci tensor read

$$\overline{\mathbf{R}}_{\hat{i}\hat{j}} = -\frac{2P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{i}}\mathring{\epsilon}_{\hat{k}\hat{n}\hat{j}}x^{\hat{m}}x^{\hat{n}}\,, \qquad \overline{\mathbf{R}}_{\hat{4}\hat{4}} = -\frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{i}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\,. \tag{3.4.12}$$

Subsequently the Ricci scalar is $\overline{\mathbf{R}} = -2P^2/r^4$.

Chapter 4

Diagrams, extensions

The aim of this chapter is to present a systematic approach to extensions of a class of spacetimes. We further introduce the conformal and projection diagrams, which are a useful tool to visualise the geometry of the extensions.

4.1 Causality for a class of bloc-diagonal metrics

We start with a construction of extensions for metrics of the form

$$g = -Fdt^2 + F^{-1}dr^2 + \underbrace{h_{AB}dx^A dx^B}_{=:h}, \qquad F = F(r)\,, \tag{4.1.1}$$

where $h := h_{AB}(t, r, x^C)dx^A dx^B$ is a family of *Riemannian* metrics on an $(n-1)$-dimensional manifold N^{n-1}, possibly depending upon t and r. Our analysis is based upon that of Walker [274].

There is a long list of important examples:

1. $F = 1 - \frac{2m}{r}$, $h = r^2 d\Omega^2$; this is the usual (3+1)-dimensional Schwarzschild solution. For $m > 0$ the function F has a simple zero at $r = 2m$.

2. $F = 1 - \frac{2m}{r^{n-1}}$, $h = r^2 d\mathring{h}$, where $\mathring{h}$ is the unit round metric on an $(n-1)$–dimensional sphere S^{n-1}; this is the $(n+1)$=dimensional Schwarzschild-Tangherlini solution. Here F has again one simple positive zero for $m > 0$.

3. $F = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$, $h = r^2 d\Omega^2$; this is the $(3+1)$-dimensional Reissner-Nordström metric, solution of the Einstein-Maxwell equations, with total electric charge Q, and with associated Maxwell potential $A = Q/r$ (compare Section 1.5. If we assume that $|Q| < m$. then F has two distinct positive zeros
$$r_\pm = m \pm \sqrt{m^2 - |Q|^2}\,,$$
and a single zero when $|Q| = m$. This last case is referred to as *extreme*, or *degenerate*;

4. $F = 1 - \frac{2m}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}}$, $h = r^2 d\Omega^2$; this is an $(n+1)$-dimensional generalization of the Reissner-Nordström metric, solution of the Einstein-Maxwell equations, with total electric charge Q, and with associated Maxwell potential $A = Q/r$;

5. $F = -\frac{\Lambda}{3}r^2 + \kappa - \frac{2m}{r}$, where Λ is the cosmological constant, and $h = r^2\mathring{h}_\kappa$, with $\mathring{h}$ having constant Gauss curvature κ:
$$\mathring{h}_\kappa = \begin{cases} d\theta^2 + \sin^2\theta d\varphi^2, & \kappa > 0; \\ d\theta^2 + d\varphi^2, & \kappa = 0; \\ d\theta^2 + \sinh^2\theta d\varphi^2, & \kappa < 0. \end{cases}$$
These are the $(3+1)$-dimensional Kottler metrics, also known as the Schwarzschild-(anti de)Sitter metrics;

6. $F(r) = 1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}$, where $\ell > 0$ is related to the cosmological constant Λ by the formula $2\Lambda = n(n-1)/\ell^2$. These are solutions of the vacuum Einstein equations if $h = r^2\mathring{h}$, provided that $\mathring{h}$ is an *Einstein metric on an $(n-1)$–dimensional manifold N^{n-1} with scalar curvature equal to* $(n-1)(n-2)$ [28]. We will refer to such metrics as *generalised Kottler metrics* or *Birmingham metrics*. Note that $m = 0$ requires $(N^{n-1}, \mathring{h})$ to be the unit round metric if one wants to avoid a singularity at finite distance, at $r = 0$, along the level sets of t.

7. We note finally the metrics
$$g = -(\lambda - \Lambda r^2)dt^2 + (\lambda - \Lambda r^2)^{-1}dr^2 + |\Lambda|^{-1}\mathring{h}_k \,, \qquad (4.1.2)$$
with $k = \pm 1$, $k\Lambda > 0$, $\lambda \in \mathbb{R}$. The case $k = 1$ has been discovered by Nariai [217].

Remark 4.1.1 It is worth pointing out that the study of the conformal structure for more general metrics of the form
$$\overset{(2)}{g} = -F(r)H_1(r)dt^2 + F^{-1}(r)H_2(r)dr^2 \,, \qquad (4.1.3)$$
where H_1 and H_2 are strictly positive in the range of r of interest, can be reduced to the one for the metric (4.1.1) by writing
$$\overset{(2)}{g} = \sqrt{H_1H_2}\left(-\hat{F}dt^2 + \hat{F}^{-1}dr^2\right) \,, \text{ where } \hat{F} = \sqrt{\frac{H_1}{H_2}}F \,. \qquad (4.1.4)$$

4.1.1 Riemannian aspects

We will be mainly interested in functions F which change sign: thus, we assume that there exists a real number r_0 such that
$$F(r_0) = 0 \,.$$
Not unexpectedly, the global properties of g will depend upon the nature of the zero of F. For example, for the Schwarzschild metric with $m > 0$ we have a first-order zero of F at $2m$. This implies that the distance of radial curves to the set $\{r = 2m\}$ is finite: indeed, letting γ denote any of the curves $r \mapsto (r, \theta, \varphi)$, the length of γ in the region $\{r > 2m\}$ is
$$\begin{aligned} \int \sqrt{g(\dot\gamma,\dot\gamma)}dr &= \int \frac{dr}{\sqrt{1-\frac{2m}{r}}} \\ &= \sqrt{r^2 - 2\,mr} + m\ln\left(r - m + \sqrt{r^2 - 2\,mr}\right) + C \,, \end{aligned}$$

where C depends upon the starting point. This has a finite limit as r approaches $2m$. More generally, if F has a first order zero, then F behaves as some constant C_1 times $(r - r_0)$ near r_0, giving a radial distance

$$\int \sqrt{g(\dot\gamma,\dot\gamma)}dr = \int \frac{1}{\sqrt{F(r)}} \approx \int \frac{1}{\sqrt{C_1(r-r_0)}} < \infty\,,$$

since $x^{-1/2}$ is integrable near $x = 0$.

On the other hand, for the *extreme* Reissner-Nordström metric we have, by definition,

$$F(r) = \Big(1 - \frac{m}{r}\Big)^2\,,$$

leading to a radial distance

$$\begin{aligned}\int \sqrt{g(\dot\gamma,\dot\gamma)}dr &= \int \frac{1}{1-\frac{m}{r}}dr = \int \frac{r}{r-m}dr = \int \frac{r-m+m}{r-m}dr \\ &= r + \ln(r-m)\,, \end{aligned} \tag{4.1.5}$$

which diverges as $r \to m$. Quite generally, if F has a zero of order $k \geq 2$ at r_0, then F behaves as some constant C_2 times $(r - r_0)^k$ near r_0, giving a radial distance

$$\int \sqrt{g(\dot\gamma,\dot\gamma)}dr = \int \frac{1}{\sqrt{F(r)}} \approx \int \frac{1}{(C_2(r-r_0))^{k/2}} \to_{r\to r_0} \infty\,,$$

since the integral of $x^{-k/2}$ near $x = 0$ diverges for $k \geq 2$.

The above considerations are closely related to the embedding diagrams, already seen in Section 1.2.6 for the Schwarzschild metric. In the Schwarzschild case those led to a hypersurface in Euclidean $\mathbb{R}^{n+1}$ which could be smoothly continued across its boundary. It is not too difficult to verify that this will be the case for any function F which has a first order zero at $r_0 > 0$.

One can likewise attempt to embed in four-dimensional Euclidean space the $t = \text{const}$ slice of the extreme Reissner-Norsdström metric. The embedding equation arising from (1.2.69) now reads

$$\Big(\frac{dz}{dr}\Big)^2 + 1 = \frac{1}{F(r)}\,, \tag{4.1.6}$$

For r close to and larger than m we obtain

$$\frac{dz}{dr} \approx \frac{1}{1-\frac{m}{r}}\,.$$

Integrating as in (4.1.5), one obtains a logarithmic divergence of the graphing function z near $r = m$:

$$z(r) \approx m\ln(r-m)\,.$$

This behaviour can also be inferred from the exact formula:

$$z = 2\sqrt{m\,(2\,r-m)} + m\ln\left(\frac{\sqrt{m\,(2\,r-m)}+m}{\sqrt{m\,(2\,r-m)}-m}\right)\,.$$

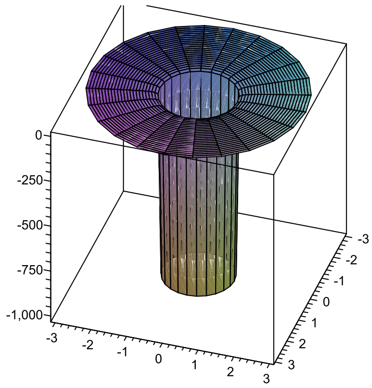

Figure 4.1.1: An isometric embedding in four-dimensional Euclidean space (one dimension suppressed) of a slice of constant time in extreme Reissner-Nordström spacetime near $r = m$.

The embedding, near $r = m$, is depicted in Figure 4.1.1

Quite generally, the behaviour near r_0 of the embedding function z, solution of (4.1.6), depends only upon the order of the zero of F at r_0. For $r_0 > 0$, and for all orders of that zero larger than or equal to two, if the "angular part" h of the metric (4.1.1) is of the form $h = r^2\mathring{h}$, where $\mathring{h}$ does not depend upon r, then the geometry of the slices $t =$ const resembles more and more that of a "cylinder" $dx^2 + r_0^2\mathring{h}$ as r_0 is approached.

4.1.2 Causality

To understand causality for metrics of the form (4.1.1), the guiding principles for the analysis that follows will be:

1. the $t - r$ part of g plays a key role;
2. multiplying the metric by a nowhere vanishing function does not matter;
3. the geometry of bounded sets is easier to visualize than that of unbounded ones.

Concerning point 1, consider a timelike curve $\gamma(s) := (t(s), r(s), x^A(s))$ for the metric (4.1.1). We have

$$\begin{aligned} 0 > g(\dot\gamma,\dot\gamma) &= -F\left(\frac{dt}{ds}\right)^2 + \frac{1}{F}\left(\frac{dt}{ds}\right)^2 + h_{AB}\left(\frac{dx^A}{ds}\right)^2\left(\frac{dx^B}{ds}\right)^2 \\ \Longrightarrow 0 > g(\dot\gamma,\dot\gamma) &= -F\left(\frac{dt}{ds}\right)^2 + \frac{1}{F}\left(\frac{dt}{ds}\right)^2 . \end{aligned} \tag{4.1.7}$$

Thus, curves which are timelike for g project to curves $(t(s), r(s))$ which are timelike for the metric

$$\overset{(2)}{g} := -Fdt^2 + \frac{1}{F}dr^2 . \tag{4.1.8}$$

Similarly, (4.1.7) shows that, for any set of constants x_0^A, a curve $(t(s), r(s))$ which is timelike for the metric $\overset{(2)}{g}$ lifts to a curve $(t(s), r(s), x_0^A)$ which is timelike for g.

Identical statements hold for causal curves.

Concerning point 2 we note that, for any positive function Ω, the sign of $g(\dot\gamma, \dot\gamma)$ is the same as that of $\Omega^2 g(\dot\gamma, \dot\gamma)$. Hence, causality for a metric g is, in many respects, identical to that of the "conformally rescaled" metric $\Omega^2 g$.

Concerning point 3, it is best to proceed via examples, presented in the next section.

4.2 The building blocs

We proceed to gather a collection of building blocs that will be used to depict the global structure of spacetimes of interest. We start with:

4.2.1 Two-dimensional Minkowski spacetime

Let $\overset{(2)}{g}$ be the two-dimensional Minkowski metric:

$$\overset{(2)}{g} = -dt^2 + dx^2 \,, \quad (t,x) \in \mathbb{R}^2 \,.$$

In order to map *conformally* $\mathbb{R}^2$ to a bounded set, we first introduce two null coordinates u and v:

$$u = t - x \,, \ v = t + x \qquad \Longleftrightarrow \qquad t = \frac{u+v}{2} \,, \ x = \frac{v-u}{2} \,. \tag{4.2.1}$$

with $\overset{(2)}{g}$ taking the form

$$\overset{(2)}{g} = -du\, dv \,.$$

We have $(t,x) \in \mathbb{R}^2$ if and only if $(u,v) \in \mathbb{R}^2$. We bring the last $\mathbb{R}^2$ to a bounded set by introducing

$$U = \arctan(u) \,, \qquad V = \arctan(v) \,, \tag{4.2.2}$$

thus

$$(U,V) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big) \times \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big) \,.$$

Using

$$\frac{du}{dU} = \frac{1}{\cos^2 U} \,, \quad \frac{dv}{dV} = \frac{1}{\cos^2 V} \,,$$

the metric becomes

$$\overset{(2)}{g} = -\frac{1}{\cos^2 U \cos^2 V} dU dV \,. \tag{4.2.3}$$

This looks somewhat more familiar if we make one last change of coordinates similar to that in (4.2.1):

$$U = T - X \,, \ V = T + X \quad \Longleftrightarrow \quad T = \frac{U+V}{2} \,, \ X = \frac{V-U}{2} \,, \tag{4.2.4}$$

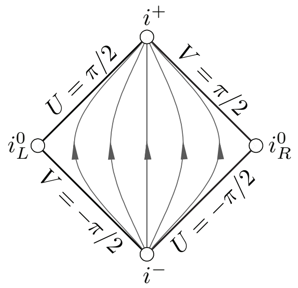

Figure 4.2.1: The conformal diagram for $(1+1)$-dimensional Minkowski space-time; see also Remark 4.2.1. I am very grateful to Christa Ölz (see [224]) and Michał Eckstein for providing the figures in this section.

leading to

$$\overset{(2)}{g} = \frac{1}{\cos^2(T-X)\cos^2(T+X)}(-dT^2+dX^2)\,.$$

We conclude that the Minkowski metric on $\mathbb{R}^2$ is conformal to the Minkowski metric on a diamond

$$\{-\pi/2 < T-X < \pi/2\,, \quad -\pi/2 < T+X < \pi/2\}\,,$$

see Figure 4.2.1.

Remark 4.2.1 Equation (4.2.3) shows that $g^{uu} = g^{vv} = 0$. This implies (see Proposition A.13.2, p. 271) that the curves $s \mapsto (u, v = s)$ are future directed null geodesics along which V approaches $\pi/2$ to the future, and $-\pi/2$ to the past. A similar observation applies to the null geodesics $s \mapsto (u = s, v)$. Thus the union of the boundary intervals

$$\mathscr{I}^- := \{U \in (-\pi/2, \pi/2)\,,\ V = -\pi/2\} \cup \{V \in (-\pi/2, \pi/2)\,,\ U = -\pi/2\}$$

can be thought of as describing initial points of null future directed geodesics; this set is usually denoted by $\mathscr{I}^-$, and is called *past null infinity*. Similarly, the set $\mathscr{I}^+$, called *future null infinity*, defined as

$$\mathscr{I}^+ := \{U \in (-\pi/2, \pi/2)\,,\ V = \pi/2\} \cup \{V \in (-\pi/2, \pi/2)\,,\ U = \pi/2\}\,,$$

is the set of end points of future directed null geodesics.

Next, every *timelike* future directed geodesic acquires an end point at

$$i^+ := (V = \pi/2, U = \pi/2)\,,$$

called *future timelike infinity*, and an initial point at

$$i^- := (V = -\pi/2, U = -\pi/2)\,,$$

called *past timelike infinity*. Finally, all spacelike geodesics accumulate at both i^0_R and i^0_L. $\square$

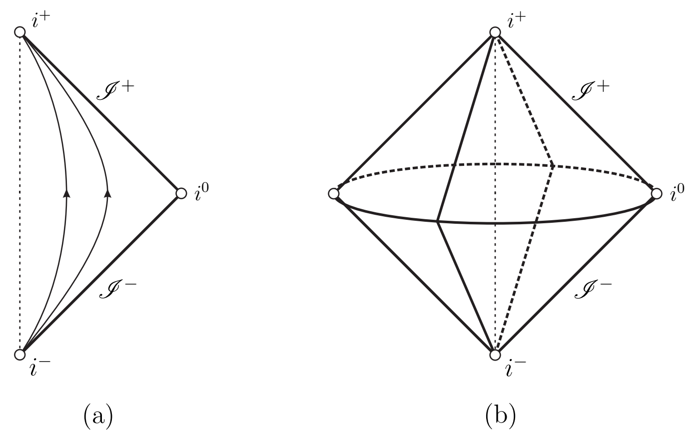

Figure 4.2.2: Conformal structure of $(1+n)$–dimensional Minkowski spacetime, $n \geq 2$ ($n = 2$ in Figure (b)); see also Figure 4.2.3.

4.2.2 Higher dimensional Minkowski spacetime

We write the $(n+1)$-dimensional Minkowski metric using spherical coordinates,

$$\eta = -dt^2 + dr^2 + r^2 d\Omega^2 \,,$$

where the symbol $d\Omega^2$ denotes the unit round metric on an $(n-1)$-dimensional sphere. In view of our principle that "for causality only the $t - r$ part of the metric matters", to understand global causality it suffices to consider the two-dimensional metric

$$\overset{(2)}{g} = -dt^2 + dr^2 \,.$$

But this is the two-dimensional Minkowski metric, so the calculations done in two dimensions apply, with x in (4.2.1) replaced by r. However, one has to keep in mind the following:

1. First, $r \equiv x \geq 0$, as opposed to $x \in \mathbb{R}$ previously. In the notation of (4.2.1)-(4.2.4) this leads to

$$\begin{aligned} x \geq 0 \quad &\Longleftrightarrow \quad v \geq u \\ &\Longleftrightarrow \quad \tan V \geq \tan U \\ &\Longleftrightarrow \quad V \geq U \\ &\Longleftrightarrow \quad X \geq 0 \,. \end{aligned}$$

   So instead of Figure 4.2.1 one obtains Figure 4.2.2(a).

2. Next, Figure 4.2.2(a) suggests that $r = 0$ is a boundary of the spacetime, which is not the case; instead it is an axis of rotation where the spheres $t = \text{const}$, $r = \text{const}$ degenerate to points. A more faithful representation in dimension 2+1 is provided by Figure 4.2.2(b). This last figure also gives an idea how the left figure should be understood in higher dimensions.

3. Finally, the conformal nature of the point i^0 of Figure 4.2.2 needs a more careful investigation: For this, let us write $R \geq 0$ for X, and return to the equations

$$\eta = -dt^2 + dr^2 + r^2 d\Omega^2 = \tfrac{1}{\cos^2(T-R)\cos^2(T+R)}(-dT^2 + dR^2) + r^2 d\Omega^2 \,,$$
$$T + R = \arctan(t + r) \,, \quad T - R = \arctan(t - r) \,.$$

Now,

$$r = \tfrac{1}{2}(\tan(R - T) + \tan(R + T)) = \tfrac{\sin(2R)}{\cos(2R)+\cos(2T)} \,,$$
$$\cos(T - R)\cos(R + T) = \tfrac{1}{2}(\cos(2R) + \cos(2T)) \,,$$

leading to

$$\eta = \frac{1}{(\cos(2R) + \cos(2T))^2}\big(\underbrace{4(-dT^2 + dR^2) + \sin^2(2R)d\Omega^2}_{=:\mathring{g}_E}\big) \,. \qquad (4.2.5)$$

(We hope that the reader will not confuse the unit round metric on S^2, usually denoted by $d\Omega^2$ in the relavistic literature, with the differential of the conformal factor Ω.) The metric

$$\mathring{g}_{S^3} := 4dR^2 + \sin^2(2R)d\Omega^2 = d\psi^2 + \sin^2(\psi)d\Omega^2 \,, \text{ where } \psi := 2R \,,$$

is readily recognized to be the unit round metric on S^n, with $R = 0$ being the north pole, and $2R = \pi$ being the south pole. Hence

$$\mathring{g}_E = -d\tau^2 + \mathring{g}_{S^3} \,, \text{ where } \tau := 2T \,,$$

is the product metric on the *Einstein cylinder* $\mathbb{R} \times S^n$. Now, for $\tau \in (-\pi, \pi)$ and $\psi \in (0, \pi)$ the condition of positivity of the conformal factor,

$$\cos(\tau) + \cos(\psi) > 0 \,,$$

is equivalent to

$$-\pi + \psi < \tau < \pi - \psi \,. \qquad (4.2.6)$$

Thus:

PROPOSITION 4.2.2 *For $n \geq 2$ the Minkowski metric is conformal to the metric on the open subset (4.2.6) of the Einstein cylinder $\mathbb{R} \times S^n$, cf. Figure 4.2.3.* □

From what has been said, it should be clear that i^0 is actually a single point in the conformally rescaled spacetime. Future null infinity $\mathscr{I}^+$ is the future light-cone of i^0 in the Einstein cylinder, and reconverges at i^+ to the past light-cone of i^+. Similarly $\mathscr{I}^-$ is the past light-cone of i^0, and reconverges at i^- to the future light-cone of i^-.

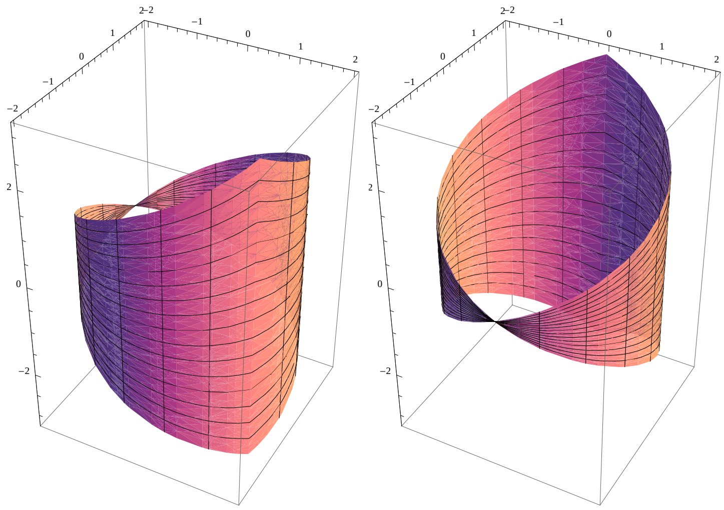

Figure 4.2.3: The embedding of Minkowski spacetime into the Einstein cylinder $\mathbb{R} \times S^3$ (two space-dimensions suppressed).

4.2.3 $\int F^{-1}$ diverging at both ends

We return now to a $\overset{(2)}{g}$ of the form (4.1.8) with non-constant F, and consider an open interval $I = (r_1, r_2)$, with $r_1 \in \mathbb{R} \cup \{-\infty\}$, $r_2 \in \mathbb{R} \cup \{\infty\}$, such that F has *constant sign on* I. We choose some $r_* \in I$, and we assume that

$$\lim_{r \to r_1} \int_r^{r_*} \frac{ds}{|F(s)|} = \infty\,, \qquad \lim_{r \to r_2} \int_{r_*}^{r} \frac{ds}{|F(s)|} = \infty\,. \tag{4.2.7}$$

Equation (4.2.7) will hold in the following cases of interest:

1. At the event horizons of all classical black holes: Schwarzschild with or without cosmological constant, Kerr-Newman with or without cosmological constant, etc. More generally, (4.2.7) will hold if $r_1 \in \mathbb{R}$ and if F extends differentiably across r_1, with $F(r_1) = 0$; note that the left integral will then diverge regardless of the order of the zero of F at r_1. A similar statement holds for r_2.

2. In the asymptotically flat regions of asymptotically flat spacetimes. Quite generally, (4.2.7) will hold if $r_2 = \infty$ and if F is bounded away from zero near r_2, as is the case for asymptotically flat regions where $F(r) \to 1$ as $r \to \infty$.

Note that the sign of F determines the causal character of the Killing vector $X := \partial_t$: X will be timelike if $F > 0$ and spacelike otherwise. Alternatively, t or $-t$ will be a time-function if $F > 0$, while r or $-r$ will be a time-function in regions where F is negative.

We introduce a new coordinate x defined as

$$x(r) = \int_{r_*}^{r} \frac{ds}{F(s)} \quad \Longrightarrow \quad dx = \frac{dr}{F(r)}\,. \tag{4.2.8}$$

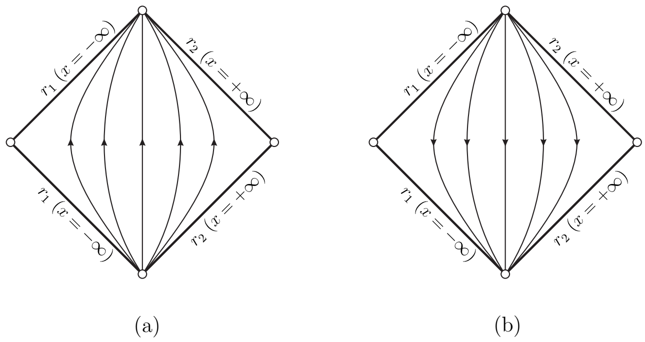

Figure 4.2.4: The conformal structure for $F > 0$ according to whether the Killing vector $X = \partial_t$ is future pointing (left) or past pointing (right). Note that in some cases r might have a wrong orientation (this occurs e.g. in the region *III* in the Kruskal-Szekeres spacetime of Figure 1.2.7), in which case one also needs to consider the mirror reflections of the above diagrams with respect to the vertical axis, compare Figures 4.2.6.

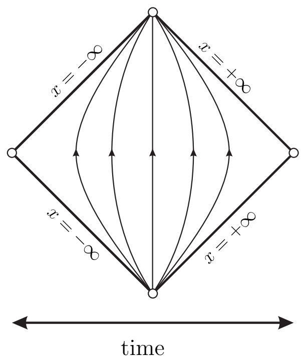

Figure 4.2.5: The conformal structure for $F < 0$.

This gives

$$\overset{(2)}{g} = -F dt^2 + \frac{1}{F}(\underbrace{dr}_{F dx})^2 = F(-dt^2 + dx^2)\,. \tag{4.2.9}$$

In view of (4.2.7) the coordinate x ranges over $\mathbb{R}$. So, *if* $F > 0$, then $\overset{(2)}{g}$ is conformal to the two-dimensional Minkowski metric, and thus the causal structure is that in Figure 4.2.4. Otherwise, for *negative* F, we obtain a Minkowski metric in which x corresponds to time and t corresponds to space, leading to a causal structure as in Figure 4.2.5. Rotating Figure 4.2.5 so that time flows to the future along the vertical positively oriented axis we obtain the four possible diagrams of Figure 4.2.6.

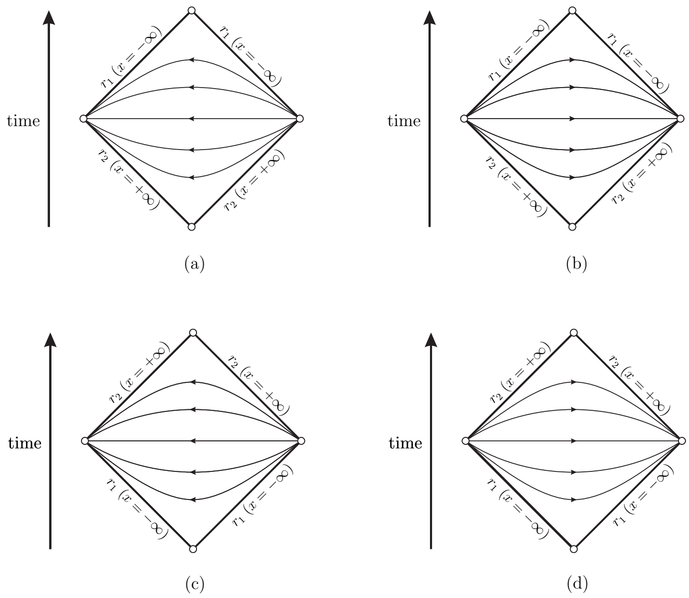

Figure 4.2.6: Figure 4.2.5 rotated so that time flows in the positive vertical direction. Four different diagrams are possible, according to whether the Killing vector $X = \partial_t$ is pointing left or right, and whether ∇r is future- or past-pointing.

4.2.4 $\int F^{-1}$ diverging at one end only

We consider again a general $\overset{(2)}{g}$ of the form (4.1.8), with F defined on an open interval $I = (r_1, r_2)$, with $r_1 \in \mathbb{R} \cup \{-\infty\}$, $r_2 \in \mathbb{R} \cup \{\infty\}$, such that F has constant sign on I. We choose some $r_* \in I$ and, instead of (4.2.7), we assume that

$$\lim_{r\to r_1} \int_r^{r_*} \frac{ds}{|F(s)|} < \infty \,, \quad \lim_{r\to r_2} \int_{r_*}^{r} \frac{ds}{|F(s)|} = \infty \,. \tag{4.2.10}$$

The case of r_1 interchanged with r_2 in (4.2.10) is analysed by replacing r by $-r$ in what follows. Note that this introduces the need of applying a mirror symmetry to the diagrams below.

The conditions in (4.2.10) arise in the following cases of interest:

1. We have $r_1 \in \mathbb{R}$, with the set $\{r = r_1\}$ corresponding either to a singularity, or to an axis of rotation. We encountered the latter possibility when analyzing $(n+1)$–dimensional Minkowki spacetime. The former situation occurs e.g. in Schwarzschild spacetime under the horizons, with $r_1 = 0$ and $r_2 = 2m$.

2. An example with $r_1 = -\infty$ is provided by anti-de Sitter spacetime, where F behaves as r^2 for large r. The variable r here should be *the negative of* the usual radial coordinate in anti-de Sitter. Yet another example of this kind occurs in the de Sitter metric, where F behaves as $-r^2$ for large $|r|$, so that r is a time function there.

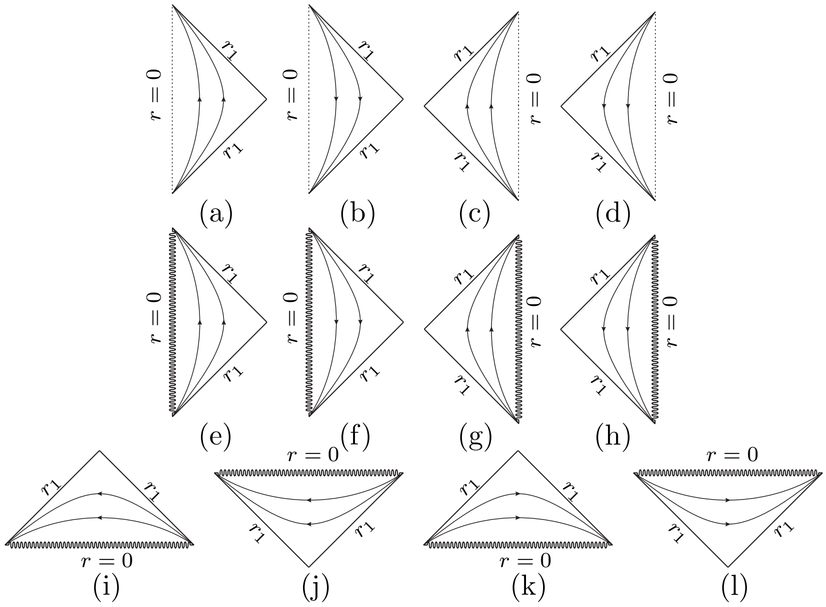

Figure 4.2.7: Some possible diagrams for (4.2.10). Time always flows forwards along the vertical axis. In (a)-(d) the set $\{r = 0\}$ corresponds to an axis of rotation; in (e)-(h) it is a singularity. There should be four more such figures where $\{r = 0\}$ should be replaced by $\{r = \infty\}$, corresponding to an asymptotic region. Similarly there should be four more figures similar to (i)-(l), where a singularity $\{r = 0\}$ is replaced by an asymptotic region $\{r = \infty\}$.

Instead of (4.2.8) we introduce a new coordinate x defined as

$$x(r) = \int_{r_1}^{r} \frac{ds}{F(s)} \,. \tag{4.2.11}$$

Equation (4.2.9) remains unchanged, but now the coordinate x ranges over $[0, \infty)$. This has already been analysed in the context higher-dimensional Minkowski spacetime, resulting in the conformal diagrams of Figure 4.2.7.

4.2.5 Generalised Kottler metrics with $\Lambda < 0$ and $m = 0$

We consider a positive function $F : [0, \infty) \to \mathbb{R}$ which has no zeros, with

$$x_\infty := \int_0^\infty \frac{dr}{F(r)} < \infty \,. \tag{4.2.12}$$

This will be e.g. the case for the generalised Kottler metrics (as described on p. 129, in the introduction to this chapter), with negative Λ, with $\kappa = 1$, and with vanishing mass $m = 0$, for which

$$F(r) = \frac{r^2}{\ell^2} + 1 \,.$$

In this case, we use (4.2.8) with $r_* = 0$, then $x \in [0, x_\infty)$; we obtain that $\overset{(2)}{g}$ is conformal to a Minkowski metric on a strip $\mathbb{R}_{\text{time}} \times [0, x_\infty)$, as in Figure 4.2.8(a).

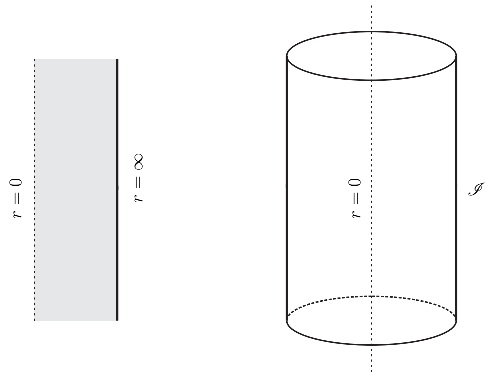

Figure 4.2.8: The conformal structure of anti-de Siter spacetime. The two-dimensional projection is the shaded strip $0 \leq r < \infty$ of figure (a). Since $\{r = 0\}$ is a center of rotation, a more faithful representation is provided by the solid cylinder of figure (b).

If the "internal space" N^{n-1} is a sphere S^{n-1}, then $\{r = 0\} \equiv \{x = 0\}$ is a rotation axis, so a more adequate representation of the resulting spacetime is provided by Figure 4.2.8(b).

4.3 Putting things together

We have now at our disposal a variety of building blocs and a natural question arises, whether or not more interesting spacetimes can be constructed using those. We start by noting that no C^2-extensions are possible across a boundary near which $|F|$ approaches infinity: Indeed, $F = -g(X, X)$, where $X = \partial_t$ is a Killing vector. Now, it is readily seen that for any Killing vector the scalar function $g(X, X)$ is bounded on compact sets, which justifies the claim. It follows that boundaries at which F becomes unbounded correspond either to a spacetime singularity, as is the case in the Schwarzschild metric at $r = 0$, or to a "boundary at infinity" representing "points lying infinitely far away".

So it remains to consider boundaries at which F tends to a finite value.

4.3.1 Four-blocs gluing

We have seen in Remark 1.2.12, p. 26, how to glue four blocs together, assuming *a first-order zero* of F. This allows us to reproduce immediately the Penrose diagram of the Schwarzschild spacetime, by gluing together across $r_1 = 2m$ two copies of bloc (a) from Figure 4.2.4 (one for which r increases from left to right, corresponding to the usual $r > 2m$ Schwarzschild region, with a mirror image thereof where r decreases from left to right), as well as blocs (i) and (j) from Figure 4.2.7.

Some further significant examples are as follows:

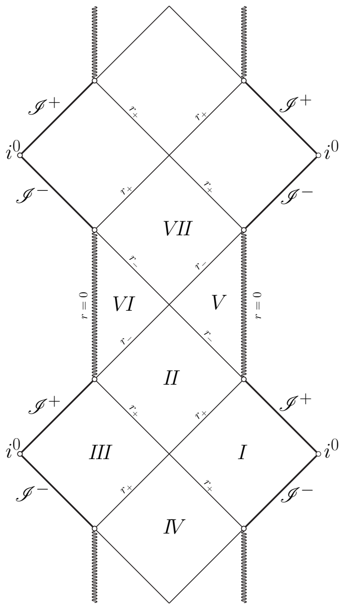

Figure 4.3.1: A maximal analytic extension for the Reissner-Nordström metric with $|Q| < m$.

Example 4.3.1 [The conformal structure of non-extreme Reissner-Nordström black holes.] Let us consider a C^k function

$$F : [0, \infty) \to \mathbb{R},$$

for some $k \geq 1$, such that F has precisely two first-order zeros at $0 < r_1 < r_2 < \infty$, and assume that

$$\lim_{r\to\infty} F(r) = 1, \quad \int_0^{\frac{r_1}{2}} \frac{dr}{F(r)} < \infty.$$

We further assume that the set $\{r = 0\}$ corresponds to a spacetime singularity. This is the behaviour exhibited by the electro-vacuum Reissner-Norsdtröm black holes with $|Q| < m$, compare Section 1.5.

One possible construction of a (maximal, analytic) extension of the region $\{r > r_2\}$ proceeds as follows: We start by noting that this region corresponds to the bloc of Figure 4.2.4(a); this is block I in Figure 4.3.1. We can perform a four-block gluing by joining together the left-right mirror image of bloc (a) from Figure 4.2.4, corresponding to the region $\{r > r_2\}$ where now r *decreases* from left to right (this is block *III* in Figure 4.3.1), as well as blocs (b) and (d) from Figure 4.2.6, corresponding to two regions $\{r_1 < r < r_2\}$. This results in the spacetime consisting of the union of blocs I to *IV* in Figure 4.3.1. This spacetime can be further extended to the future, via a four-blocs gluing, by adding two triangles (c) and (g) from Figure 4.2.7, and yet another region $\{r_1 < r < r_2\}$ from Figure 4.2.6. This leads to a spacetime consisting of the union of blocs I to *VII* in Figure 4.3.1. One can now continue periodically in

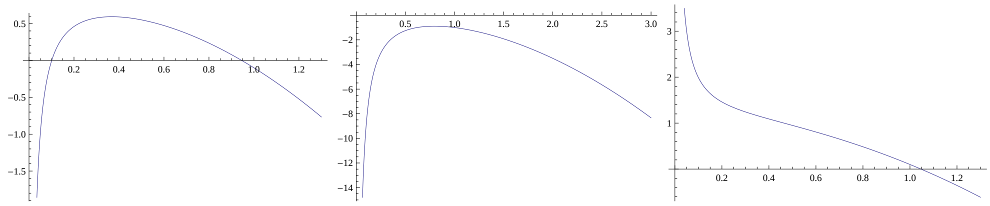

Figure 4.3.2: The function F when (from left to right) a) m is positive but smaller than the threshold of (4.3.2), b) m is positive and larger than the threshold, and c) m is negative.

time, both to the future and to the past, obtaining the infinite sequence of blocs of Figure 4.3.1.

Note that identifying periodically in time, with distinct periods, provides a countable infinity of distinct alternative extensions. The resulting spacetimes contain closed timelike curves, and no black hole region.

Further maximal analytic distinct extensions can be obtained by removing a certain number of bifurcation spheres from the spacetime depicted in Figure 4.3.1, and passing to the universal cover of the resulting spacetime. There are then no causality violations, as opposed to the examples of the previous paragraph. On the other hand the current construction leads to spacetimes containing incomplete geodesics on which e.g. the norm of the Killing vector ∂_t remains bounded, while no such geodesics exist in the spacetimes of the previous paragraph.

EXAMPLE 4.3.2 ("SCHWARZCHILD-DE SITTER" METRICS.) We consider a function $F \in C^k([0,\infty))$, $k \geq 1$, which has precisely two first-order zeros at $0 < r_1 < r_2 < \infty$, which is negative for large r, and which satisfies

$$\int_{2r_2}^{\infty} \frac{dr}{|F(r)|} < \infty \,.$$

We again assume that the set $\{r = 0\}$ corresponds to a spacetime singularity. This is the behaviour of the "generalized Kottler [174] metrics", also known as "Birmingham [28] metrics" in $n+1$ dimensions (cf. Section 4.6, p. 150), with cosmological constant $\Lambda > 0$, under suitable restrictions on m: The metric takes the form

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)}dr^2 + r^2\mathring{h}, \quad \text{where} \quad F(r) = 1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}\,, \quad (4.3.1)$$

where $\ell > 0$ is related to the cosmological constant Λ by the formula $2\Lambda = n(n-1)/\ell^2$, while $\mathring{h}$ denotes an Einstein metric with constant scalar curvature $(n-1)(n-2)$ on a manifold N^{n-1}. Representative graphs of the function F are shown in Figure 4.3.2. We will only consider the case $m > 0$ and

$$\left(\frac{2}{(n-1)(n-2)}\right)^{n-2} \Lambda^{n-2} m^2 n^2 < 1\,, \quad (4.3.2)$$

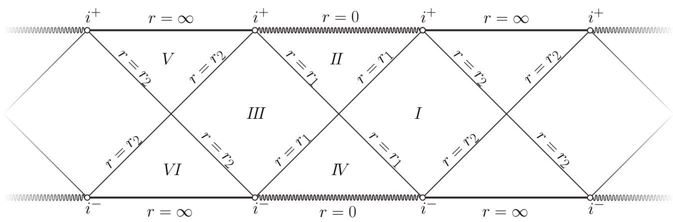

Figure 4.3.3: A maximal extension of the class (4.3.2) of generalized Kottler (Schwarzchild – de Sitter) metrics with positive cosmological constant and mass.

which are sufficient and necessary conditions for exactly two distinct positive first-order zeros of F. The remaining cases are discussed in Section 4.6 below. When $n = 3$, the condition (4.3.2) reads $9m^2\Lambda < 1$, and the case of equality is referred to as the extreme Kottler–Schwarzschild–de Sitter spacetime. In the limit where Λ tends to zero with m held constant, the spacetime metric approaches the Schwarzschild metric with mass m, and in the limit where m goes to zero with Λ held constant the metric tends to that of the de Sitter spacetime with cosmological constant Λ.

To obtain a maximal extension, as $F(r)$ is positive for $r \in (r_1, r_2)$ we can choose, say, bloc (a) of Figure 4.2.4 as the starting point of the construction; this is bloc I of Figure 4.3.3. A four-bloc gluing can then be done using further the mirror reflection of bloc (b) of Figure 4.2.4 as well as blocs (j) and (k) of Figure 4.2.7 to extend I to the spacetime consisting of blocs *I-IV* of Figure 4.3.3. Continuing similarly across $r = r_2$, etc., one obtains the infinite sequence of blocs of Figure 4.3.3.

Let us denote by $(\mathscr{M}, g)$ the spacetime constructed as in Figure 4.3.3, then $\mathscr{M}$ is diffeomorphic to $\mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{space}} \times N^{n-1}$. Note that Figure 4.3.3 remains unchanged when shifted by two blocs to the left or right. This leads to a discrete isometry of the associated spacetime $(\mathscr{M}, g)$, let's call it ψ. Given $k \in \mathbb{N}$, one can then consider the quotient manifold $\mathscr{M}/\psi^k$, with the obvious metric. This is the same as introducing periodic identifications in Figure 4.3.3, identifying a bloc with its image obtained by shifting by a multiple of $2k$ blocs to the left or to the right. The resulting spacetime will have topology $\mathbb{R} \times S^1 \times N^{n-1}$, in particular it will contain *compact* spacelike hypersurfaces, with topology $S^1 \times N^{n-1}$. For distinct k's the resulting spacetimes will be diffeomorphic, but not isometric.

Example 4.3.3 [Kottler/de Sitter metrics with positive cosmological constant, and vanishing mass parameter m.] We consider the metrics (4.3.1) with $m = 0$, thus $F(r) = 1 - r^2/\ell^2$. Then F has one simple zero for positive r. By arguments already given above one is led to the conformal diagram of Figure 4.3.4.

Example 4.3.4 [Kottler/anti de Sitter metrics with negative cosmological constant.] The reader should have no difficulties to show that

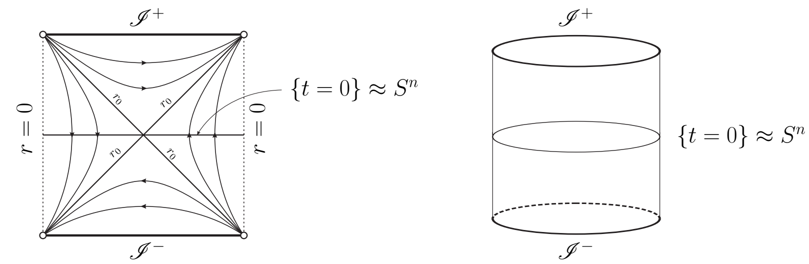

Figure 4.3.4: The generalized Kottler (de Sitter) metrics with positive cosmological constant and vanishing mass parameter m. Left figure: a conformal diagram; the lines $\{r = 0\}$ are centers of rotation. The right figure makes it clearer that the Cauchy surface $\{t = 0\}$, as well as $\mathscr{I}^+$ and $\mathscr{I}^-$, have spherical topology.

the metrics (4.3.1) with $\Lambda < 0$ and $m \neq 0$ can be extended to a spacetime as in Figure 4.7.12, p. 187 below, without however the shaded region there as there are no time-machines in the solution when the angular-momentum parameter a vanishes.

EXAMPLE 4.3.5 [NARIAI METRICS WITH $\lambda\Lambda > 0$.] The Nariai metrics can be written in the form

$$g \quad = \quad -(\lambda - \Lambda r^2)dt^2 + (\lambda - \Lambda r^2)^{-1}dr^2 + |\Lambda|^{-1}\mathring{h}_k \;, \tag{4.3.3}$$

with constants satisfying $k = \pm 1$, $k\Lambda > 0$, $\lambda \in \mathbb{R}$. The metric g will satisfy the Lorentzian $(n+1)$-dimensional vacuum Einstein equations with cosmological constant proportional to Λ (equal to Λ in spacetime dimension four) if and only if $\mathring{h}_k$ is a Riemannian Einstein metric on a $(n-1)$-dimensional manifold, say N, with scalar curvature equal to a suitable, dimension dependent constant, whose sign coincides with that of k.

The lapse function g_{tt} has two first order zeros $r_- < r_+$ if and only if $\lambda\Lambda > 0$, with the Killing vector ∂_t timelike between r_- and r_+; in all remaining cases $\lambda\Lambda \leq 0$ we obtain directly an inextendible spacetime, without Killing horizons, and thus a somewhat dull product structure. When $r \to \pm\infty$ we have $|g_{tt}| \to \infty$, and since the norm of a Killing vector is a geometric invariant, no extension is possible there. One can then obtain a global extension shown in Figure 4.3.5.

The case $\lambda\Lambda > 0$ but $\Lambda < 0$ leads to a global structure described by rotating Figure 4.3.5 by 90 degrees.

For further reference we note alternative forms of g. When $\lambda > 0$ and $\Lambda > 0$, a constant rescaling of t and r leads to

$$g = \Lambda^{-1}\left(-(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + \mathring{h}_k\right) . \tag{4.3.4}$$

In the region $r^2 < 1$ (regions I and III in Figure 4.3.5) we can set $r = \cos(x)$,

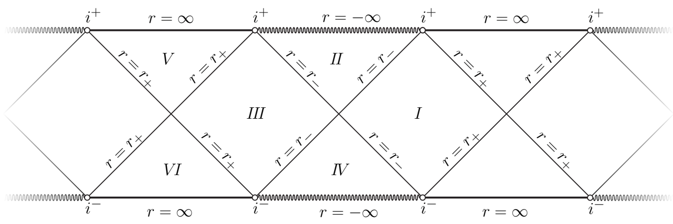

Figure 4.3.5: A maximal extension of Nariai metrics with $\lambda > 0$ and $\Lambda > 0$.

so that

$$g = \Lambda^{-1}\left(-\sin^2(x)dt^2 + dx^2 + \mathring{h}_k\right) . \tag{4.3.5}$$

In the region $r^2 > 1$ (regions II, IV, V and VI in Figure 4.3.5) we can set $r = \cosh(\tau)$ and $y = t$ in (4.3.4), which results in

$$g = \Lambda^{-1}\left(-d\tau^2 + \sinh^2(\tau)dy^2 + \mathring{h}_k\right) . \tag{4.3.6}$$

In either case, the space-part of the metric has cylindrical structure, with a product metric on $\mathbb{R} \times N$.

Amusingly, the metric (4.3.6) can be obtained from (4.3.5) by replacing $x \mapsto i\tau$ and $t \mapsto y$. Further complex substitutions in (4.3.6), namely $\tau \mapsto \tau + i\pi/2$ and $y \mapsto iy$, lead to the metric

$$g = \Lambda^{-1}\left(-d\tau^2 + \cosh^2(\tau)dy^2 + \mathring{h}_k\right) , \tag{4.3.7}$$

with cylindrical spatial slices and boring global structure.

When λ and Λ are both negative, a constant rescaling of t and r leads instead to

$$g = |\Lambda|^{-1}\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + \mathring{h}_k\right) , \tag{4.3.8}$$

subsequently leading to obvious sign changes in (4.3.5)-(4.3.7).

4.3.2 Two-blocs gluing

The four-blocs gluing construction requires a first order zero of F; but there exist metrics of interest where F has zeros of order two. Examples are provided by the extreme Reissner-Nordström metrics, with $|Q| = m$, or by the extreme generalized Kottler metrics, for which the inequality in (4.3.2) is an equality. In such cases a two-bloc gluing applies, which works regardless of the order of the zero of F, and which proceeds as follows: Consider a function F defined on an interval I, which might or might not change sign on I, with one single zero there. As in (1.2.44) of Remark 1.2.12, p. 26, we introduce functions u and v defined as

$$u = t - f(r) , \quad v = t + f(r) , \quad f' = \frac{1}{F} . \tag{4.3.9}$$

But now one does *not* use u and v simultaneously; instead one considers, first, a coordinate system (u, r), so that

$$\overset{(2)}{g} = -F(\underbrace{dt}_{du+\frac{1}{F}dr})^2 + \frac{1}{F}dr^2 = -Fdu^2 - 2du\,dr\,.$$

Since $\det \overset{(2)}{g} = -1$, the resulting metric extends smoothly as a Lorentzian metric to the whole interval of definition, say I, of F. If we further replace u by a coordinate $U = \arctan u$, as in (4.2.2), each level set of U is extended from its initial range $r \in (r_1, r_2)$ to the whole range I. In terms of the blocs of, say, Figure 4.2.4, this provides a way of extending across the lower-left interval $r = r_1$ and/or across the upper-right interval $r = r_2$; similarly for the remaining blocs. Equivalently, the (u, r)-coordinates allow one to attach another bloc from our collection at the boundaries $V = -\pi/2$ and $V = \pi/2$, with $V = \arctan v$ as in (4.2.2).

Next, using (v, r) as coordinates one obtains

$$\overset{(2)}{g} = -F(\underbrace{dt}_{dv-\frac{1}{F}dr})^2 + \frac{1}{F}dr^2 = -Fdv^2 + 2dv\,dr\,.$$

The (V, r)-coordinates provide a way of extending across the boundary intervals $U = \pm\pi/2$; in Figure 4.2.4 these are the (open) lower-right or upper-left boundary intervals.

Example 4.3.6 [The global structure of extreme Reissner-Nordström black holes $|Q| = m$.] For extreme Reissner-Nordström metrics the function F equals

$$F(r) = \left(1 - \frac{m}{r}\right)^2 .$$

Although this is not needed for our purposes here, we note the explicit form of the function f in (4.3.9):

$$f(r) = r - \frac{m^2}{r - m} + 2\,m \ln |r - m|\ .$$

To construct a maximal extension of the exterior region $r \in (m, \infty)$, we start with a bloc (a) from Figure 4.2.4 with $r \in (m, \infty)$; this is region *I* in Figure 4.3.6. This can be extended across the upper-left interval with a bloc (e) from Figure 4.2.7, providing region *II* in Figure 4.3.6. That last bloc can be extended by another bloc identical to *I*. Continuing in this way leads to the infinite sequence of blocs of Figure 4.3.6.

4.4 General rules

For definiteness we will assume in this section that the full spacetime metric takes the form

$$-F(r)dt^2 + \frac{dr^2}{F(r)} + \underbrace{r^2\mathring{h}_{AB}(x^C)dx^A dx^B}_{=:h}, \qquad (4.4.1)$$

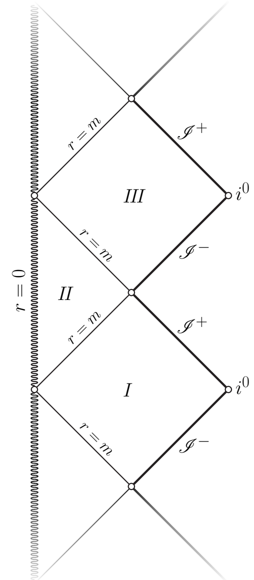

Figure 4.3.6: A maximally extended extreme Reissner-Nordström spacetime.

which holds for many metrics of interest. The reader should have no difficulties adapting the discussion here to more general metrics, e.g. such as in Section 4.7 below. The function r in (4.4.1) will be referred to as the *radius function*.

Our constructions so far lead to the following picture: Consider any two blocs from the collection provided so far, with corresponding functions F_1 and F_2, and Killing vectors X_1 and X_2 generating translations in the coordinate t of (4.4.1). Then we have the following rules:

1. Before any gluing all two-dimensional coordinate-domains should be viewed as open subsets of the plane, without their boundaries.

2. Two such blocs can be attached together across an *open* boundary interval to obtain a metric of class C^k if the corresponding radius functions take the same finite value at the boundary, and if the function F_2 extends F_1 in a C^k way across the boundary. One might sometimes have to change the space-orientation $x \to -x$ of one of the blocs to achieve this, and perhaps also the time orientation so that the (extended) Killing vector X_1 matches X_2 at the relevant boundary interval.

3. The calculation in Example 1.3.9, p. 41, shows that the surface gravity of a horizon $r = r_*$, where $F(r_*) = 0$, for metrics of the form (4.4.1) equals $F'(r_*)/2$. Hence, in view of what has just been said, a necessary and sufficient condition for a C^1 gluing of the metric across an open boundary interval is equality of the *radius functions* r together with equality of the *surface gravities* of the Killing vectors ∂_t at the boundary in question.

4. Two-bloc gluings only attach the common *open* boundary interval to the existing structure, so that the result is again an open subset of $\mathbb{R}^2$. In

particular two-bloc gluings never attach the corners of the blocs to the spacetime.

5. Four-bloc gluings can only be done across a first-order zero of F at which F is differentiable. A function F which is of C^k-differentiability class leads then to a metric which is of C^{k-1}-differentiability class.

6. A four-bloc gluing attaches to the spacetime the common corner of the four-blocs, as well as the four open intervals accumulating at the corner. (The result is of course again an open subset of $\mathbb{R}^2$.)

4.5 Black holes / white holes

One of the points of the conformal diagrams above is, that one can by *visual inspection* decide whether or not a spacetime, constructed by the prescription just given, contains a black hole region. The key observation is, that each boundary which is represented by a line of 45-degrees slope corresponds to a null hypersurface in spacetime. If the spacetime is faithfully represented by a collection of blocks on the plane, the corresponding hypersurfaces are two-sided in spacetime, and can therefore

only be crossed by future directed causal curves from one side to the other.

So consider a spacetime which contains a block with a boundary which has a slope of either 45 or -45 degrees. Let Γ denote a straight line in the plane which contains that boundary. Assuming the usual time orientation, it should be clear that no future directed causal curve with initial point in that part of the plane which lies above Γ will ever reach that part of the plane which lies under Γ. In other words, the region above Γ is inaccessible to any observer that remains entirely under Γ.

We conclude that if a *physically preferred* bloc lies under Γ, then anything above Γ will belong to a *black-hole region*, as defined relatively to that block.

One can similarly talk about *white hole regions*, by reversing time-orientation.

As an example consider bloc I of Figure 4.3.1, p. 142, describing one connected component of the infinitely many "exterior", $r > r_+$, regions of a maximally extended non-degenerate Reissner-Nordström solution. Everything lying above the line of 45-degrees slope bounding this bloc belongs to a black hole region, as defined with respect to this block. Everything lying below the line of minus 45-degrees slope bounding this bloc belongs to a white hole region, as defined with respect to block I.

Note that this argument might fail if the spacetime is *not* faithfully represented by a subset of the plane, for example if some identifications between various blocks are made, as already mentioned at the end of Example 4.3.1, p. 141.

4.6 Birmingham metrics

Further examples of interesting causal diagrams can be constructed for the *Birmingham metrics* [28], which are higher-dimensional generalisations of the Schwarzschild metric, including a non-vanishing cosmological constant. The object of what follows is to discuss those metrics, with particular emphasis on the global structure of extensions and their causal diagrams.

Consider an $(n+1)$-dimensional metric, $n \geq 3$, of the form

$$g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \underbrace{\breve{h}_{AB}(x^C)dx^A dx^B}_{=:\breve{h}}, \tag{4.6.1}$$

where $\breve{h}$ is a *Riemannian Einstein metric* on a compact $(n-1)$-dimensional manifold N, and where we denote by x^A the local coordinates on N. As first pointed out by Birmingham in [28], for any $m \in \mathbb{R}$ and

$$\ell \in \mathbb{R}^* \cup \sqrt{-1}\mathbb{R}^*$$

the function

$$f = \frac{\breve{R}}{(n-1)(n-2)} - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}, \tag{4.6.2}$$

where $\breve{R}$ is the (constant) scalar curvature of $\breve{h}$, leads to a vacuum metric,

$$R_{\mu\nu} = \frac{n}{\ell^2} g_{\mu\nu}, \tag{4.6.3}$$

where ℓ is a constant related to the cosmological constant $\Lambda \in \mathbb{R}$ as

$$\frac{1}{\ell^2} = \frac{2\Lambda}{n(n-1)}. \tag{4.6.4}$$

A comment about negative Λ, and thus purely complex ℓ's, is in order. In this section we will be mostly interested in a positive cosmological constant, which corresponds to real ℓ. When considering a negative cosmological constant (4.6.4) requires $\ell \in \sqrt{-1}\mathbb{R}$, which is awkward to work with. So when $\Lambda < 0$ it is convenient to change r^2/ℓ^2 in f to $-r^2/\ell^2$, change the sign in (4.6.4), and use a real ℓ. We will often do this without further ado.

Clearly, n cannot be equal to two in (4.6.2), and we therefore exclude this dimension in what follows.

The multiplicative factor two in front of m is convenient in dimension three when $\breve{h}$ is a unit round metric on S^2, and we will keep this form regardless of topology and dimension of N.

There is a rescaling of the coordinate $r = b\bar{r}$, with $b \in \mathbb{R}^*$, which leaves (4.6.1)-(4.6.2) unchanged if moreover

$$\bar{\breve{h}} = b^2 \breve{h}, \quad \bar{m} = b^{-n} m, \quad \bar{t} = bt. \tag{4.6.5}$$

We can use this to achieve

$$\beta := \frac{\breve{R}}{(n-1)(n-2)} \in \{0, \pm 1\}. \tag{4.6.6}$$

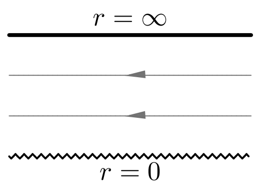

Figure 4.6.1: The (t, r)-causal diagram when $m < 0$ and f has no zeros.

This will be assumed from now on. The set $\{r = 0\}$ corresponds to a singularity when $m \neq 0$. Except in the case $m = 0$ and $\beta = -1$, by an appropriate choice of the sign of b we can always achieve $r > 0$ in the regions of interest. This will also be assumed from now on.

For reasons which should be clear from the main text, we are seeking functions f which, after a suitable extension of the spacetime manifold and metric, lead to spatially periodic solutions.

4.6.1 Cylindrical solutions

Consider, first, the case where f has no zeros. Since f is negative for large $|r|$, f is negative everywhere. It therefore makes sense to rename r to $\tau > 0$, t to x, and $-f$ to $F > 0$, leading to the metric

$$g = -\frac{d\tau^2}{F(\tau)} + F(\tau)dx^2 + \tau^2 \mathring{h} \,. \tag{4.6.7}$$

The non-zero level-sets of the time coordinate τ are infinite cylinders with topology $\mathbb{R} \times \mathring{M}$, with a product metric. Note that the extrinsic curvature of those level sets is never zero because of the r^2 term in front of $\mathring{h}$, except possibly for the $\{r = 0\}$-slice in the case $\beta = -1$ and $m = 0$.

Assuming that $m \neq 0$, the region $r \equiv \tau \in (0, \infty)$ is a "big-bang – big freeze" spacetime with cylindrical spatial sections. The corresponding Penrose diagram is an infinite horizontal strip with a singular spacelike boundary at $\tau = 0$, and a smooth conformal spacelike boundary at $\tau = \infty$, see Figure 4.6.1.

In the case $m = 0$ and $\beta = 0$ the spatial sections are again cylindrical, with the boundary $\{\tau = 0\}$ being now at infinite temporal distance: Indeed, setting $T = \ln \tau$, in this case we can write

$$\begin{aligned} g &= -\ell^2 \frac{d\tau^2}{\tau^2} + \frac{\tau^2}{\ell^2} dx^2 + \tau^2 \mathring{h} \\ &= -\ell^2 dT^2 + e^{2T} \left(\frac{dx^2}{\ell^2} + \mathring{h} \right) . \end{aligned}$$

When $\mathring{h}$ is a flat torus, this is one of the forms of the de Sitter metric [147, p. 125].

The next case which we consider is $f \leq 0$, with f vanishing precisely at one positive value $r = r_0$. This occurs if and only if $\beta = 1$ and

$$r_0 = \sqrt{\frac{n}{n-2}}\,\ell \,, \quad m = \frac{r_0^n}{(n-2)\ell^2} \,. \tag{4.6.8}$$

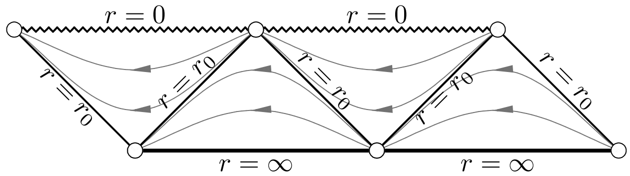

Figure 4.6.2: The causal diagram for the Birmingham metrics with positive cosmological constant and $f \leq 0$, vanishing precisely at r_0.

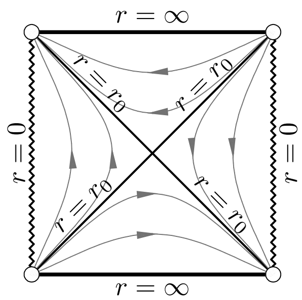

Figure 4.6.3: The causal diagram for the Birmingham metrics with positive cosmological constant and $m < 0$, $\beta \in \mathbb{R}$, or $m = 0$ and $\beta = 1$, with r_0 defined by the condition $f(r_0) = 0$. The set $\{r = 0\}$ is a singularity unless the metric is the de Sitter metric ($\mathring{M} = S^{n-1}$ and $m = 0$), or a suitable quotient thereof so that $\{r = 0\}$ corresponds to a center of (possibly local) rotational symmetry.

A $(r = \tau, t = x)$-causal diagram can be found in Figure 4.6.2.

No non-trivial, periodic, time-symmetric ($K_{ij} = 0$) spacelike hypersurfaces occur in all spacetimes above. Periodic spacelike hypersurfaces with $K_{ij} \not\equiv 0$ arise, but a Hamiltonian analysis of initial data asymptotic to such hypersurfaces goes beyond the scope of this work.

From now on we assume that f has positive zeros.

4.6.2 Naked singularities

Assuming that $m = 0$ but $\beta \neq 0$, we must have $\beta = 1$ in view of our hypothesis that f has positive zeros. For $r \geq 0$ the function f has exactly one zero, $r = \ell$. The boundaries $\{r = 0\}$ and $\{r = \ell\}$ of the set $\{r \in [0, \ell]\}$ correspond either to regular centers of symmetry, in which case the level sets of t are S^n's or their quotients, or to conical singularities. See Figure 4.6.3.

If $m < 0$ the function $f : (0, \infty) \to \mathbb{R}$ is monotonously decreasing, tending to minus infinity as r tends to zero, where a naked singularity occurs, and to minus infinity when r tends to ∞, hence f has then precisely one zero. The causal diagram can be seen in Figure 4.6.3.

No spatially periodic time-symmetric spacelike hypersurfaces occur in the spacetimes above.

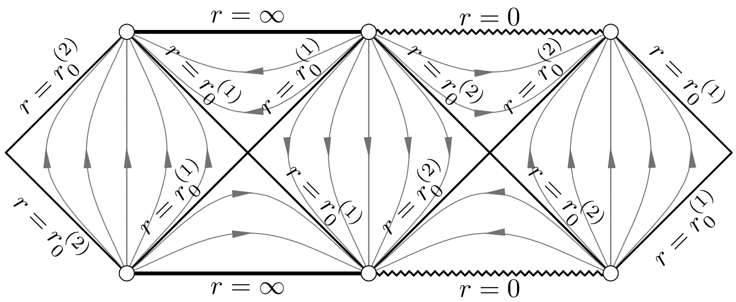

Figure 4.6.4: The causal diagram for Birmingham metrics with $\Lambda > 0$ and exactly two first-order zeros of f.

4.6.3 Spatially periodic time-symmetric initial data

We continue with the remaining cases, that is, f having zeros and $m > 0$. The function $f : (0,\infty) \to \mathbb{R}$ is then concave and thus has precisely two first order zeros, except for the case already discussed in (4.6.8). A causal diagram for a maximal extension of the spacetime, for the two-first-order-zeros cases, is provided by Figure 4.6.4. The level sets of t within each of the diamonds in that figure can be smoothly continued across the bifurcation surfaces of the Killing horizons to smooth spatially-periodic Cauchy surfaces.

4.6.4 Killing horizons

The locations of Killing horizons of the Birmingham metrics are defined, in space-dimension n, by the condition

$$f(r_0) = \beta - \frac{2m}{r_0^{n-1}} - \frac{r_0^2}{\ell^2} = 0\,.$$

Thus, variations of the metric on the horizons satisfy

$$0 = \delta f|_{r=r_0} = \left[(\partial_r f)\delta r - \frac{2}{r^{n-2}}\delta m\right]\Big|_{r=r_0}; \tag{4.6.9}$$

equivalently

$$\delta m = \frac{1}{2(n-1)}(\partial_r f)\delta(r^{n-1}) = \frac{1}{(n-1)\sigma_{n-1}}\frac{(\partial_r f)}{2}\Big|_{r=r_0}\delta A\,, \tag{4.6.10}$$

where $r^{n-1}\sigma_{n-1}$ is the "area" of the cross-section of the horizon.

Let us check that $\kappa := \frac{(\partial_r f)}{2}\Big|_{r=r_0}$ coincides with the surface gravity of the horizon, defined through the usual formula

$$\nabla_K K = -\kappa K\,, \tag{4.6.11}$$

where K is the Killing vector field which is null on the horizon. For this, we rewrite the spacetime metric (4.6.1) as usual as:

$$g = -f\,du^2 - 2\,du\,dr + r^2\mathring{h}\,,$$

where $du = dt - \frac{1}{f}\,dr$. The Killing field $K = \partial_u = \partial_t$ is indeed tangent to the horizon and null on it. Formula (4.6.11) implies that

$$\kappa = -\Gamma^u_{uu} = -\frac{1}{2}g^{u\lambda}(2g_{\lambda u,u} - g_{uu,\lambda})\,. \tag{4.6.12}$$

The inverse metric equals

$$g^\sharp = -2\frac{\partial}{\partial u}\frac{\partial}{\partial r} + f\left(\frac{\partial}{\partial r}\right)^2 + r^{-2}\mathring{h}^\sharp\,,$$

whence $g^{u\lambda} = -\delta^\lambda_r$, and

$$\kappa = -\frac{1}{2}g_{uu,r} = \frac{(\partial_r f)}{2}\Big|_{r=r_0}\,,$$

as claimed. We conclude that on Killing horizons it holds that

$$\delta m = \frac{1}{(n-1)\sigma_{n-1}}\kappa\Big|_{r=r_0}\delta A\,. \tag{4.6.13}$$

Equation (4.6.13) if often referred to as the *first law of black hole dynamics.*

4.6.5 Curvature

In this section we study the geometry of metrics of the form

$$g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\underbrace{\mathring{h}_{AB}(x^C)dx^A dx^B}_{=:\mathring{h}} \tag{4.6.14}$$

in the region where $f < 0$. For visual clarity it is convenient to make the following replacements and redefinitions:

$$r \to \tau\,, \quad t \to x\,, \quad f \to -e^{2\chi}\,, \tag{4.6.15}$$

which bring g to the form

$$g = -e^{-2\chi(\tau)}d\tau^2 + e^{2\chi(\tau)}dx^2 + \tau^2\mathring{h}\,, \tag{4.6.16}$$

To calculate the Riemann tensor we use the moving-frames formalism. For $A = 1,\ldots,n$ let $\mathring{\theta}^A$ be an ON-coframe for $\mathring{h}$,

$$\mathring{h} = \sum_{A=1}^{n-1}\mathring{\theta}^A\otimes\mathring{\theta}^A\,,$$

and let $\mathring{\omega}_{AB}$ and $\mathring{\Omega}_{AB}$ be the associated connection and curvature forms. It holds that

$$\begin{aligned} 0 &= d\mathring{\theta}^A + \mathring{\omega}^A{}_B\wedge\mathring{\theta}^B\,,\\ \mathring{\Omega}^A{}_B &= d\mathring{\omega}^A{}_B + \mathring{\omega}^A{}_C\wedge\mathring{\omega}^C{}_B\,. \end{aligned}$$

Let θ^μ be the following g-ON-coframe:

$$\theta^0 = e^{-\chi} d\tau \,, \quad \theta^A = \tau \mathring{\theta}^A \,, \quad \theta^n = e^{\chi} dx \,.$$

The vanishing of torsion gives

$$\begin{aligned}
0 &= d\theta^0 + \omega^0{}_\mu \wedge \theta^\mu = \omega^0{}_n \wedge \theta^n + \omega^0{}_A \wedge \theta^A \,, \\
0 &= d\theta^A + \omega^A{}_\mu \wedge \theta^\mu = d\tau \wedge \mathring{\theta}^A + \tau d\mathring{\theta}^A + \omega^A{}_\mu \wedge \theta^\mu \\
&= d\tau \wedge \mathring{\theta}^A + \omega^A{}_0 \wedge \theta^0 + \omega^A{}_n \wedge \theta^n + (\omega^A{}_B - \mathring{\omega}^A{}_B) \wedge \theta^B \,, \\
0 &= d\theta^n + \omega^n{}_\mu \wedge \theta^\mu = d(e^\chi) \wedge dx + \omega^n{}_\mu \wedge \theta^\mu \\
&= e^\chi \dot{\chi}\, \theta^0 \wedge \theta^n + \omega^n{}_0 \wedge \theta^0 + \omega^n{}_A \wedge \theta^A \,.
\end{aligned}$$

This is solved by setting

$$\begin{aligned}
\omega^n{}_A &= 0 \,, \\
\omega^n{}_0 &= e^\chi \dot{\chi}\, \theta^n = \frac{1}{2}(\dot{e^{2\chi}}) dx \,, \\
\omega^A{}_0 &= e^\chi \mathring{\theta}^A \,, \\
\omega^A{}_B &= \mathring{\omega}^A{}_B \,.
\end{aligned}$$

The curvature two-forms are thus

$$\begin{aligned}
\Omega^0{}_n &= d\omega^0{}_n + \omega^0{}_\mu \wedge \omega^\mu{}_n = d\omega^0{}_n = \frac{1}{2}(\ddot{e^{2\chi}}) d\tau \wedge dx \\
&= \frac{1}{2}(\ddot{e^{2\chi}})\, \theta^0 \wedge \theta^n = \frac{1}{2}(\ddot{e^{2\chi}}) \delta^0_{[\mu} g_{\nu]n} \theta^\mu \wedge \theta^\nu \,, &(4.6.17) \\
\Omega^0{}_A &= d\omega^0{}_A + \omega^0{}_\mu \wedge \omega^\mu{}_A = \frac{1}{2}(\dot{e^{2\chi}}) \theta^0 \wedge \mathring{\theta}_A + e^\chi d\theta_A + e^\chi \theta_B \wedge \omega^B{}_A \\
&= \frac{1}{2}(\dot{e^{2\chi}}) \tau^{-1} \theta^0 \wedge \theta_A = \frac{1}{2}(\dot{e^{2\chi}}) \tau^{-1} \delta^0_{[\mu} g_{\nu]A} \theta^\mu \wedge \theta^\nu \,, &(4.6.18) \\
\Omega^n{}_A &= d\omega^n{}_A + \omega^n{}_\mu \wedge \omega^\mu{}_A = \frac{1}{2}(\dot{e^{2\chi}}) \tau^{-1} \theta^n \wedge \theta_A \\
&= \frac{1}{2}(\dot{e^{2\chi}}) \tau^{-1} \delta^n_{[\mu} g_{\nu]A} \theta^\mu \wedge \theta^\nu \,, &(4.6.19) \\
\Omega^A{}_B &= d\omega^A{}_B + \omega^A{}_\mu \wedge \omega^\mu{}_B = \mathring{\Omega}^A{}_B + e^{2\chi} \tau^{-2} \theta^A \wedge \theta_B \\
&= \frac{1}{2} \mathring{\Omega}^A{}_{BCD} \mathring{\theta}^C \wedge \mathring{\theta}^D + e^{2\chi} \tau^{-2} \theta^A \wedge \theta_B \\
&= \frac{1}{2} \tau^{-2} (\mathring{\Omega}^A{}_{BCD} + 2 e^{2\chi} \delta^A_{[C} \delta_{D]B})\, \theta^C \wedge \theta^D \,. &(4.6.20)
\end{aligned}$$

Using

$$\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} \theta^\alpha \wedge \theta^\beta \,, \tag{4.6.21}$$

we conclude that, up to symmetries, the non-zero frame-components of the Riemann tensor are

$$\begin{aligned}
R^0{}_{n\mu\nu} &= (\ddot{e^{2\chi}}) \delta^0_{[\mu} g_{\nu]n} \,, &(4.6.22) \\
R^0{}_{A\mu\nu} &= (\dot{e^{2\chi}}) \tau^{-1} \delta^0_{[\mu} g_{\nu]A} \,, &(4.6.23) \\
R^n{}_{A\mu\nu} &= (\dot{e^{2\chi}}) \tau^{-1} \delta^n_{[\mu} g_{\nu]A} \,, &(4.6.24) \\
R^A{}_{BCD} &= \tau^{-2} (\mathring{\Omega}^A{}_{BCD} + 2 e^{2\chi} \delta^A_{[C} \delta_{D]B}) \,. &(4.6.25)
\end{aligned}$$

Hence the non-vanishing components of the Ricci tensor are

$$R_{00} = -\frac{1}{2}\Big((\ddot{e^{2\chi}}) + (n-1)(\dot{e^{2\chi}})\tau^{-1}\Big) = -R_{nn}\,, \tag{4.6.26}$$

$$R_{AB} = \tau^{-2}\mathring{R}_{AB} + \Big((n-2)\tau^{-2}e^{2\chi} + (\dot{e^{2\chi}})\tau^{-1}\Big)\,g_{AB}\,. \tag{4.6.27}$$

If $\mathring{h}$ is Einstein, $\mathring{R}_{AB} = (\mathring{R}/(n-1))\mathring{h}_{AB}$, the last equation becomes

$$R_{AB} = \tau^{-2}\bigg(\frac{\mathring{R}}{n-1} + (n-2)e^{2\chi} + \tau(\dot{e^{2\chi}})\bigg)g_{AB}\,. \tag{4.6.28}$$

It is now straightforward to check that for any $m \in \mathbb{R}$ and $\ell \in \mathbb{R}^*$ the function

$$e^{2\chi} = -\frac{\mathring{R}}{(n-1)(n-2)} + \frac{2m}{\tau^{n-2}} + \frac{\tau^2}{\ell^2} \tag{4.6.29}$$

(compare with (4.6.2) and (4.6.14)-(4.6.15)) leads to a vacuum metric:

$$R_{\mu\nu} = \frac{n}{\ell^2}g_{\mu\nu}\,. \tag{4.6.30}$$

For further reference we note that the Ricci scalar R equals, quite generally,

$$R = (\ddot{e^{2\chi}}) + (n-1)\Big(2(\dot{e^{2\chi}})\tau^{-1} + (n-2)\tau^{-2}e^{2\chi}\Big) + \tau^{-2}\mathring{R}. \tag{4.6.31}$$

Suppose that g is a Birmingham metric with $m = 0$, thus

$$e^{2\chi} = -\beta + \frac{\tau^2}{\ell^2}$$

for a constant β, then

$$\frac{1}{2}(\ddot{e^{2\chi}}) = \frac{1}{2}(\dot{e^{2\chi}})\tau^{-1} = \tau^{-2}(e^{2\chi} + \beta) = \frac{1}{\ell^2}\,.$$

If $\mathring{h}$ is a space-form, with

$$\mathring{\Omega}^A{}_{BCD} = 2\beta\delta^A_{[C}\delta_{D]B}\,,$$

consistently with (4.6.6), we obtain

$$R_{\mu\nu\rho\sigma} = \frac{2}{\ell^2}g_{\mu[\rho}g_{\sigma]\nu}\,.$$

If, however, $\mathring{h}$ is *not* a space-form, we have

$$\mathring{\Omega}^A{}_{BCD} = 2\beta\delta^A_{[C}\delta_{D]B} + r^A{}_{BCD}\,,$$

for some non-identically vanishing tensor $r^A{}_{BCD}$, with all traces zero. Hence we obtain

$$R_{\mu\nu\rho\sigma} = \frac{2}{\ell^2}g_{\mu[\rho}g_{\sigma]\nu} + \tau^{-2}r_{\mu\nu\rho\sigma}\,,$$

where the functions $r_{\mu\nu\rho\sigma}$ are τ-independent in the current frame, and vanish whenever one of the indices is 0 or n. This gives

$$\begin{aligned} R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} &= \frac{2n(n+1)}{\ell^4} + r^{\mu\nu\rho\sigma}r_{\mu\nu\rho\sigma} \\ &= \frac{2n(n+1)}{\ell^4} + \tau^{-4} \sum_{A,B,C,D=1}^{n-1} (r_{ABCD})^2 \,, \end{aligned}$$

which is singular at $\tau = 0$.

Recall, now, that the calculations so far also apply to

$$g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\mathring{h} \tag{4.6.32}$$

with the following replacements and redefinitions:

$$r \to \tau\,, \quad t \to x\,, \quad f \to -e^{2\chi}\,. \tag{4.6.33}$$

Strictly speaking, f should be negative when using the substitutions above, but the final formulae hold regardless of the sign of f. For convenience of crossreferencing we rewrite the formulae obtained so far in this notation:

$$R^r{}_{t\mu\nu} = -f''\delta^r_{[\mu}g_{\nu]t}\,, \tag{4.6.34}$$
$$R^r{}_{A\mu\nu} = -f'r^{-1}\delta^r_{[\mu}g_{\nu]A}\,, \tag{4.6.35}$$
$$R^t{}_{A\mu\nu} = -f'r^{-1}\delta^t_{[\mu}g_{\nu]A}\,, \tag{4.6.36}$$
$$R^A{}_{BCD} = r^{-2}(\mathring{\Omega}^A{}_{BCD} - 2f\delta^A_{[C}\delta_{D]B})\,, \tag{4.6.37}$$
$$R_{rt} = 0 = R_{tA} = R_{rA}\,, \tag{4.6.38}$$
$$R_{rr} = \frac{1}{2}\left(f'' + (n-1)f'r^{-1}\right) = -R_{tt}\,, \tag{4.6.39}$$
$$R_{AB} = r^{-2}\mathring{R}_{AB} - \left((n-2)r^{-2}f + f'r^{-1}\right)g_{AB}\,. \tag{4.6.40}$$

$$R = -f'' - (n-1)\left(2f'r^{-1} + (n-2)r^{-2}f\right) + r^{-2}\mathring{R}\,. \tag{4.6.41}$$

If $\mathring{h}$ is Einstein

$$\mathring{R}_{AB} = \frac{\mathring{R}}{n-1}\mathring{h}_{AB}\,, \tag{4.6.42}$$

the last equation becomes

$$R_{AB} = r^{-2}\left(\frac{\mathring{R}}{n-1} - (n-2)f - rf'\right)g_{AB}\,. \tag{4.6.43}$$

As before, for any $m \in \mathbb{R}$ and $\ell \in \mathbb{R}^*$ the function

$$f = \frac{\mathring{R}}{(n-1)(n-2)} - \frac{2m}{r^{n-2}} - \varepsilon\frac{r^2}{\ell^2}\,, \qquad \varepsilon \in \{0, \pm 1\} \tag{4.6.44}$$

leads to an Einstein metric:

$$R_{\mu\nu} = \varepsilon\frac{n}{\ell^2}g_{\mu\nu}\,. \tag{4.6.45}$$

4.6.6 The Euclidean Schwarzschild - anti de Sitter metric

An important role in Euclidean quantum gravity [137] is played by solutions of the field equations with Riemannian signature. An example of such a metric is provided by the *Euclidean Schwarzschild anti-de Sitter metric* which, in $(n+1)$-dimensions, takes the form

$$g = \underbrace{\left(\frac{r^2}{\ell^2} + \kappa - \frac{2m}{r^{n-2}}\right)}_{=:F(r)} dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + \kappa - \frac{2m}{r^{n-2}}} + r^2 h_\kappa \,, \tag{4.6.46}$$

where $\ell > 0$ and m are real constants, $\kappa \in \{0, \pm 1\}$, and where $({}^{n-1}N, h_\kappa)$ is an $(n-1)$-dimensional Einstein manifold with Ricci tensor equal to $(n-2)\kappa h_\kappa$.

The metric (4.6.46) is obtained from the Schwarzschild - anti-de Sitter metric by replacing dt^2 by $-dt^2$. Such a substitution preserves the condition that the Ricci tensor is proportional to the metric, which can be seen as follows:

Quite generally, let g be a metric such that $\partial_t g_{\mu\nu} = 0$ in a suitable coordinate system. Consider the tensor field, say $g(a)$, where every occurrence of dt in g is replaced by adt, where $a \in \mathbb{C}$. Let $R_{\mu\nu}(a)$ denote the Ricci tensor of $g(a)$. Then $R_{\mu\nu}(a) - \lambda g_{\mu\nu}(a)$ is a holomorphic function of a away from the set where $\det g(a)_{\mu\nu}$ vanishes. When $a \in \mathbb{R}^+$ the metric $g(a)$ can be obtained from g by a coordinate transformation $t \mapsto at$, hence $R_{\mu\nu}(a) - \lambda g_{\mu\nu}(a)$ with $a \in \mathbb{R}$ vanishes if

$$R_{\mu\nu}(1) - \lambda g_{\mu\nu}(1) = R_{\mu\nu} - \lambda g_{\mu\nu}$$

did. Since a holomorphic function vanishing on the real positive axis vanishes everywhere, we conclude that $R_{\mu\nu}(a) - \lambda g(a) = 0$ for all a on the connected component of $\mathbb{C}$ containing 1 on which $\det g(a)_{\mu\nu} \neq 0$.

We conclude that g given by (4.6.46) is indeed an Einstein metric.

Let $r_* > 0$ be any *first-order* zero of g_{tt},

$$\frac{r_*^2}{\ell^2} + \kappa - \frac{2m}{r_*^{n-2}} = 0\,.$$

After introducing a new coordinate ρ by the formula

$$\rho(r) = \int_{r_*}^{r} \frac{1}{\sqrt{\frac{s^2}{\ell^2} + \kappa - \frac{2m}{s^{n-2}}}} ds\,, \tag{4.6.47}$$

one can rewrite the metric (4.6.46) as

$$g = d\rho^2 + \rho^2 H(\rho) dt^2 + r^2 h_\kappa\,, \tag{4.6.48}$$

where H is obtained by dividing g_{tt} by ρ^2. Elementary analysis, using the fact that r_* is a simple zero of F, shows that

$$H(0) = \frac{F'(r_*)^2}{4}\,.$$

This implies that a periodic identification of t with period

$$T := \frac{4\pi}{F'(r_*)}$$

guarantees that $d\rho^2 + \rho^2 H(\rho)dt^2$ is a smooth metric on $\mathbb{R}^2$ with a rotation axis at $\rho = 0$. As a result, (4.6.48) defines a smooth Riemannian metric on

$$M := \mathbb{R}^2 \times {}^{n-1}N\,.$$

The metric (4.6.46) can be smoothly conformally compactified by introducing, for large r, a coordinate $x := 1/r$ and rescaling:

$$x^2 g = (\frac{1}{\ell^2} + \kappa - 2mx^n)dt^2 + \frac{dx^2}{\frac{1}{\ell^2} + \kappa x^2 - 2mx^n} + h_\kappa\,. \tag{4.6.49}$$

Hence, the conformal boundary $\partial M := \{x = 0\}$ of M is diffeomorphic to $S^1 \times {}^{n-1}N$, with conformal metric

$$\frac{dt^2}{\ell^2} + h_\kappa\,. \tag{4.6.50}$$

Horowitz-Myers-type metrics

Consider an $(n+1)$-dimensional metric, $n \geq 3$, of the form

$$g = f(r)d\psi^2 + \frac{dr^2}{f(r)} + r^2 \underbrace{\check{h}_{AB}(x^C)dx^A dx^B}_{=:\check{h}} \tag{4.6.51}$$

where now $\check{h}$ is a Riemannian *or* pseudo-Riemannian Einstein metric on an $(n-1)$-dimensional manifold $\mathring{N}$ with constant scalar curvature $\mathring{R}$ and, similarly to the last section, the x^A's are local coordinates on $\mathring{N}$.[1] This metric can be formally obtained from (4.6.1) by changing t to $i\psi$. It therefore follows that for $m \in \mathbb{R}$ and $\ell \in \mathbb{R}^*$ the function

$$f = \beta - \frac{2m}{r^{n-2}} - \varepsilon\frac{r^2}{\ell^2}\,, \quad \varepsilon \in \{0, \pm 1\}\,, \quad \beta = \frac{\mathring{R}}{(n-1)(n-2)} \in \{0, \pm 1\}\,. \tag{4.6.52}$$

leads to a metric satisfying (4.6.3).

$$R_{\mu\nu} = \frac{\varepsilon n}{\ell^2} g_{\mu\nu}\,, \quad \varepsilon \in \{0, \pm 1\}\,, \tag{4.6.53}$$

where ℓ is a constant related to the cosmological constant as in (4.6.4).

Suppose that f has zeros, and let us denote by r_0 the largest zero of f. We assume that r_0 is of first order, and we restrict attention to $r \geq r_0$. Imposing a suitable ψ_0-periodicity condition on $\psi \in [0, \psi_0]$, the usual arguments imply that the set $\{r = r_0\}$ is a rotation axis in a plane on which $\sqrt{r - r_0}$ and ψ are coordinates of polar type: Indeed, if we set

$$\rho = F(r)\,, \text{ with } F = \int_{r_0}^r \frac{1}{\sqrt{f(r)}} dr = \frac{\sqrt{r - r_0}}{2\sqrt{f'(r_0)}}(1 + O(r - r_0))\,,$$

[1]To avoid a proliferation of notation we use the symbol $\check{h}$ both for the metric on N appearing in (4.6.1) and for the metric on the manifold $\mathring{N}$ relevant for (4.6.51). Typically $(N, \check{h})$ is a compact Riemannian manifold, while $(\mathring{N}, \check{h})$ in (4.6.51) will be Lorentzian with $\mathring{N}$ non-compact.

we find

$$\frac{dr^2}{f} + f d\psi^2 = d\rho^2 + f(F^{-1}(\rho))d\psi^2 = d\rho^2 + (2f'(r_0))^2(1 + O(\rho^2))\rho^2 d\psi^2 \,,$$

which defines a smooth metric near $\rho = 0$ if and only if

$$\psi = \lambda \ell \alpha \,, \tag{4.6.54}$$

where α is a new 2π-periodic coordinate, and

$$\lambda = \frac{1}{2\ell f'(r_0)} \,. \tag{4.6.55}$$

In the case where

$$\varepsilon = -1 \,,$$

one obtains Einstein metrics with a negative cosmological constant.

Whatever ε, a conformal completion at spacelike infinity can be obtained by introducing a new coordinate $x = \ell/r$, bringing g to the form

$$\begin{aligned} g &= f(\ell x^{-1})\ell^2\lambda^2 d\alpha^2 + \frac{\ell^2 dx^2}{x^4 f(\ell x^{-1})} + \ell^2 x^{-2}\breve{h} \\ &= x^{-2}\ell^2\big(-(\varepsilon - \beta x^2 + O(x^n))\lambda^2 d\alpha^2 - (\varepsilon + \beta x^2 + O(x^n))dx^2 + \breve{h}\big) \end{aligned} \tag{4.6.56}$$

We see explicitly that the conformal class of metrics induced by $x^2 g$ on the boundary at infinity,

$$\mathscr{I} = \{x = 0\} \approx S^1 \times \mathring{N} \,,$$

is Lorentzian if $\breve{h}$ is Lorentzian and if $\varepsilon = -1$.

$\beta = 0$, $n = 3$

In [156] Horowitz and Myers consider the case $n + 1 = 4$, $\varepsilon = -1$,[2] and choose $\breve{h} = -\ell^{-2}dt^2 + d\varphi^2$, with φ being a 2π-periodic coordinate on S^1. Thus

$$g = -\frac{r^2}{\ell^2}dt^2 + f(r)\ell^2\lambda^2 d\alpha^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2 \,. \tag{4.6.57}$$

Equation (4.6.56) shows that timelike infinity $\mathscr{I} \approx \mathbb{R} \times S^1 \times S^1$ is conformally flat:

$$x^2 g \to_{r\to\infty} -dt^2 + \ell^2(\lambda^2 d\alpha^2 + dx^2 + d\varphi^2) \,. \tag{4.6.58}$$

Some comments about factors of ℓ are in order: if we think of r as having dimension of length, then ℓ, t and ψ also have dimension of length, m has dimension length^{n-1}, while f, x, and the x^A's (and thus φ) are dimensionless.

A uniqueness theorem for the metrics (4.6.57) has been established in [279].

[2] The case $\beta = 0$ and $\varepsilon = 1$ leads to a signature $(+---)$ for large r; our signature $(-+++)$ is recovered by multiplying the metric by minus one, but then one is back in the case $\varepsilon = -1$ after renaming m to $-m$.

$\beta = \pm 1$, $n = 3$

We consider the metric (4.6.51) with[2] $\varepsilon = -1$ and $\check h$ of the form

$$\check h = \begin{cases} d\theta^2 + \sin^2(\theta)\, d\varphi^2, & \beta = 1; \\ d\theta^2 + \sinh^2(\theta)\, d\varphi^2, & \beta = -1. \end{cases} \tag{4.6.59}$$

In regions where f is positive, one obtains a Lorentzian metric after a "double Wick rotation"

$$\theta = i\ell^{-1}t\,,\ \varphi = i\phi\,,$$

resulting in

$$g = -\frac{r^2}{\ell^2}dt^2 + \frac{dr^2}{f(r)} + f(r)\ell^2\lambda^2 d\alpha^2 + r^2 \begin{cases} \sinh^2(\ell^{-1}t)\, d\phi^2, & \beta = 1; \\ \sin^2(\ell^{-1}t)\, d\phi^2, & \beta = -1. \end{cases} \tag{4.6.60}$$

Taking α and ϕ periodic one obtains again a conformal infinity diffeomorphic to $\mathbb{R} \times \mathbb{T}^2$. Note that the conformal metric at the conformal boundary is *not* conformally stationary anymore, as opposed to the metrics (4.6.58). We have not attempted to study the nature of the singularities of g at $\sinh(\ell^{-1}t) = 0$ or at $\sin(\ell^{-1}t) = 0$.

Negative coordinate mass

For completeness we show that the metric (4.6.51) has the striking property that its total coordinate mass is negative when m is positive; the latter is needed for regularity of the metric. This has already been observed in [156] in space-dimension three with a toroidal Scri. Here we check that this remains correct in higher dimensions, for a large class of topologies of Scri.

Before continuing, we note that Lorentzian Horowitz-Myers-type metrics with a smooth conformal compactification at infinity exist only with negative Λ: Indeed, to obtain the right signature for large r when $\epsilon > 0$ one needs to multiply the metric by minus one. But then the resulting metric has negative Ricci scalar, and hence solves Einstein equations with a negative cosmological constant.

Somewhat more generally, consider those metrics of the form (4.6.51) for which

$$\mathring{\check N} = \mathbb{R}_t \times \check N\,,$$

where $(\check N, \check h)$ is a compact Riemannian manifold, and where

$$\check h = -\ell^{-2}dt^2 + \check h\,, \tag{4.6.61}$$

so that

$$g = f(r)d\psi^2 + \frac{dr^2}{f(r)} + r^2\left(-\ell^{-2}dt^2 + \check h\right)\,. \tag{4.6.62}$$

The question arises, how to define the mass of such a metric.

To avoid ambiguities, let us write f_m for the function f of (4.6.52).

Let us denote by f_m the function f of (4.6.52). One assigns a *coordinate mass* to a metric such as (4.6.62) by writing it in the form (4.6.1)-(4.6.2), p. 150, with some function f_M, then the parameter M is, by definition, the coordinate mass.

For this we introduce in (4.6.62) a new coordinate $r = r(\rho)$:

$$\begin{aligned} g &= f_m(r)\ell^2\lambda^2 d\alpha^2 + \frac{dr^2}{f_m(r)} + \frac{r^2}{\ell^2}(-dt^2 + \ell^2\check{h}) \\ &= -\frac{r^2}{\ell^2}dt^2 + \left(\frac{dr}{d\rho}\right)^2 \frac{d\rho^2}{f_m(r)} \\ &\quad +r^2((1 + O(\beta r^{-2}) + O(mr^{-n}))\lambda^2 d\alpha^2 + \check{h}) \,, \end{aligned} \tag{4.6.63}$$

where the error terms have to be understood for large r. We will have

$$g \approx -f_M(\rho)dt^2 + \frac{d\rho^2}{f_M(\rho)} + \rho^2(\lambda^2 d\alpha^2 + \check{h}) \,,$$

for some parameter M possibly different from m, provided that

$$r^2 = \ell^2 f_M(\rho)(1 + o(\rho^{-n})) \,, \quad \left(\frac{d\rho}{dr}\right)^2 f_m(r) = f_M(\rho)(1 + o(\rho^{-n})) \,, \tag{4.6.64}$$

The first equation determines r as a function of ρ up to correction terms $o(\rho^{-n})$. Inserting the result into the second equation determines M, provided that the asymptotic expansion of the left-hand side is compatible with that of the right-hand side.

Now, it is straightforward to check that these equations are compatible if and only if

$$\beta = 0 \,. \tag{4.6.65}$$

We conclude that for metrics satisfying (4.6.51)-(4.6.52) and (4.6.61)

the coordinate mass is only defined if $\beta = 0$.

Assuming (4.6.65), after asymptotically solving the first equation in (4.6.64) and inserting the result into the second one, we find that

$$\rho = r + \frac{\ell^2 M}{r^{n-1}} + O(r^{-(2n-1)}) \,, \tag{4.6.66}$$

and that the coordinate mass equals

$$M = -\frac{m}{n-1} \,. \tag{4.6.67}$$

In particular M is negative for positive m.

4.7 Projection diagrams

We have seen that a very useful tool for visualizing the geometry of two-dimensional Lorentzian manifolds is that of *conformal Carter-Penrose diagrams*. For spherically symmetric geometries, or more generally for metrics in bloc-diagonal form, the two-dimensional conformal diagrams provide useful information about the four-dimensional geometry as well, since many essential aspects of the spacetime geometry are described by the $t-r$ sector of the metric.

The question then arises, whether some similar device can be used for metrics which are not in bloc-diagonal form. In the following sections we show, following closely [87], that one can usefully represent classes of non-spherically symmetric geometries in terms of two-dimensional diagrams, called *projection diagrams*, using an auxiliary two-dimensional metric constructed out of the spacetime metric. Whenever such a construction can be carried-out, the issues such as stable causality, global hyperbolicity, existence of event or Cauchy horizons, the causal nature of boundaries, and existence of conformally smooth infinities become evident by inspection of the diagrams, in a way completely analogous to the bloc-diagonal case.

4.7.1 The definition

Let $(\mathscr{M}, g)$ be a smooth spacetime, and let $\mathbb{R}^{1,n}$ denote the $(n+1)$-dimensional Minkowski spacetime. We wish to construct a map π from $(\mathscr{M}, g)$ to $\mathbb{R}^{1,1}$ which allows one to obtain information about the causality properties of $(\mathscr{M}, g)$. Ideally, π should be defined and differentiable throughout $\mathscr{M}$. However, already the example of Minkowski spacetime, discussed in Section 4.2.2, p. 135, shows that such a requirement is too restrictive: the map used there is not differentiable at the axis of rotation. So, while we will require that π is defined everywhere, it will be convenient to require that π be differentiable, and a submersion, on a subset of $\mathscr{M}$ which we will denote by $\mathscr{U}$. (Recall that π is a submersion if π_* is surjective at every point.) This allows us to talk about "the projection diagram of Minkowski spacetime", or "the projection diagram of Kerr spacetime", rather than of "the projection diagram of the subset $\mathscr{U}$ of Minkowski spacetime", etc. Note that the latter terminology would be more precise, and will sometimes be used, but appears to be an overkill in most cases.

Now, to preserve causality it appears a good idea to map timelike vectors to timelike vectors. This will be part of our definition: π will be required to have this property on $\mathscr{U}$. But note that a necessary condition for existence of a map from $\mathscr{M}$ to $\mathbb{R}^{1,1}$ which maps timelike vectors to timelike vectors is *stable causality* of $\mathscr{U}$: Indeed, if t is a time function on $\mathbb{R}^{1,1}$, then $t \circ \pi$ will be a time function on $\mathscr{U}$ for such maps; but the existence of a time function on $\mathscr{U}$ is precisely the definition of stable causality. So causality violations provide an obvious obstruction for the construction of π.

Having accepted that $\mathscr{U}$ might not be the whole of $\mathscr{M}$, a possible requirement could be that $\mathscr{U}$ is dense in $\mathscr{M}$, as is the case for Minkowski spacetime. Keeping in mind that the Kerr spacetime contains causality-violating regions, which obviously have to be excluded from the domain where π has good causality properties, we see that the density requirement cannot be imposed in general. Clearly one would like $\mathscr{U}$ to be as large as possible: the larger $\mathscr{U}$, the more information we will get about $\mathscr{M}$. We leave it as an open question, whether or not there is an optimal largeness condition which could be imposed on $\mathscr{U}$. We simply use $\mathscr{U}$ as part of the input data of the definition, hoping secretly that it is as large as can be.

As already mentioned, we will require timelike vectors to be mapped to timelike vectors. Note that if some timelike vectors in the image of π within Minkowski spacetime will not arise as projections of timelike vectors, then there will be Minkowskian timelike curves in the image of π which will have nothing to do with causal curves in $\mathscr{M}$. But then no much insight into the causality of $\mathscr{M}$ will be gained by inspecting causal curves in $\mathbb{R}^{1,1}$. In order to avoid this, one is finally led to the following definition:

Definition 4.7.1 *Let $(\mathscr{M}, g)$ be a Lorentzian manifold. A* projection diagram *is a pair $(\pi, \mathscr{U})$, where*

$$\mathscr{U} \subset \mathscr{M}\,,$$

is open and non-empty, and where

$$\pi : \mathscr{M} \to \mathscr{W}$$

is a continuous map, differentiable on an open dense subset of $\mathscr{M}$, such that $\pi|_{\mathscr{U}}$ is a smooth submersion. Moreover:

1. *for every smooth timelike curve $\sigma \subset \pi(\mathscr{U})$ there exists a smooth timelike curve γ in $(\mathscr{U}, g)$ such that σ is the projection of γ: $\sigma = \pi \circ \gamma$;*

2. *the image $\pi \circ \gamma$ of every smooth timelike curve $\gamma \subset \mathscr{U}$ is a timelike curve in $\mathbb{R}^{1,1}$.*

Some further comments are in order:

First, we have assumed for simplicity that $(\mathscr{M}, g)$, $\pi|_{\mathscr{U}}$, and the causal curves in the definition are smooth, though assuming that π is C^1 on $\mathscr{U}$ would suffice for most purposes.

As already discussed, the requirement that timelike curves in $\pi(\mathscr{U})$ arise as projections of timelike curves in $\mathscr{M}$ ensures that causal relations on $\pi(\mathscr{U})$, which can be seen by inspection of $\pi(\mathscr{U})$, reflect causal relations on $\mathscr{M}$. Conditions 1 and 2 taken together ensure that causality on $\pi(\mathscr{U})$ represents as accurately as possible causality on $\mathscr{U}$.

The second condition of the definition is of course equivalent to the requirement that the images by π_* of timelike vectors in $T\mathscr{U}$ are timelike. This implies further that the images by π_* of causal vectors in $T\mathscr{U}$ are causal. But it should be kept in mind that projections lose information, so that the images by π_* of many null vectors in $T\mathscr{U}$ will be timelike. And, of course, many spacelike vectors will be mapped to causal vectors under π_*.

The curve-equivalent of the last remarks is that images of causal curves in $\mathscr{U}$ are causal in $\pi(\mathscr{U})$; that many spacelike curves in $\mathscr{U}$ will be mapped to causal curves in $\pi(\mathscr{U})$; and that many null curves in $\mathscr{U}$ will be mapped to timelike ones in $\pi(\mathscr{U})$.

The requirement that π is a submersion guarantees that open sets are mapped to open sets. This, in turn, ensures that projection diagrams with the same set $\mathscr{U}$ are locally unique, up to a local conformal isometry of two-dimensional Minkowski spacetime. We do not know whether or not two surjective projection diagrams $\pi_i : \mathscr{U} \to \mathscr{W}_i$, $i = 1, 2$, with identical domain of definition $\mathscr{U}$ are (globally) unique, up to a conformal isometry of $\mathscr{W}_1$ and $\mathscr{W}_2$. It would be of interest to settle this question.

In many examples of interest the set $\mathscr{U}$ will *not* be connected; we will see that this happens already in the Kerr spacetime.

Recall that a map is proper if inverse images of compact sets are compact. In the definition we could further have required π to be proper; indeed, many projection diagrams below have this property. This is actually useful, as then the inverse images of globally hyperbolic subsets of $\mathscr{W}$ are globally hyperbolic, and so global hyperbolicity, or lack thereof, can be established by visual inspection of $\mathscr{W}$. It appears, however, more convenient to talk about *proper projection diagrams* whenever π is proper, allowing for non-properness in general.

As such, we have assumed for simplicity that π maps $\mathscr{M}$ into a subset of Minkowski spacetime. In some applications it might be natural to consider more general two-dimensional manifolds as the target of π; this requires only a trivial modification of the definition. An example is provided by the Gowdy metrics on a torus, discussed at the end of this section, where the natural image manifold for π is $(-\infty, 0) \times S^1$, equipped with a flat product metric. Similarly, maximal extensions of the class of Kerr-Newman - de Sitter metrics of Figure 4.7.8, p. 182, require the image of π to be a suitable Riemann surface.

4.7.2 Simplest examples

The simplest examples of projection diagrams have already been constructed for metrics of the form

$$g = e^f(-Fdt^2 + F^{-1}dr^2) + \underbrace{h_{AB}dx^A dx^B}_{=:h}, \qquad F = F(r), \tag{4.7.1}$$

where $h = h_{AB}(t, r, x^C)dx^A dx^B$ is a family of *Riemannian* metrics on an $(n-1)$-dimensional manifold N^{n-1}, possibly depending upon t and r, and f is a function which is allowed to depend upon all variables. It should be clear that any manifestly conformally flat representation of any extension, defined on $\mathscr{W} \subset \mathbb{R}^{1,1}$, of the two-dimensional metric $-Fdt^2 + F^{-1}dr^2$, as discussed in Section 4.3, provides immediately a projection diagram for $(\mathscr{W} \times N^{n-1}, g)$.

In particular, introducing spherical coordinates (t, r, x^A) on

$$\mathscr{U} := \{(t, \vec{x}) \in \mathbb{R}^{n+1}, |\vec{x}| \neq 0\} \subset \mathbb{R}^{1,n} \tag{4.7.2}$$

and forgetting about the $(n-1)$-sphere-part of the metric leads to a projection diagram for Minkowski spacetime which coincides with the usual conformal diagram of the fixed-angles subsets of Minkowski spacetime (see the left figure in Figure 4.2.2, p. 135). The set $\mathscr{U}$ defined in (4.7.2) cannot be extended to include the world-line passing through the origin of $\mathbb{R}^n$ since the map π fails to be differentiable there. This diagram is proper, but fails to represent correctly the nature of the spacetime near the set $|\vec{x}| = 0$.

On the other hand, a globally defined projection diagram for Minkowski spacetime (thus, $(\mathscr{U}, g) = \mathbb{R}^{1,n}$) can be obtained by writing $\mathbb{R}^{1,n}$ as a product $\mathbb{R}^{1,1} \times \mathbb{R}^{n-1}$, and forgetting about the second factor. This leads to a projection

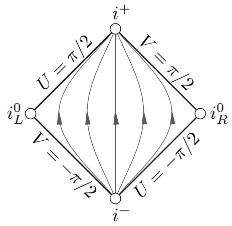

Figure 4.7.1: The conformal diagram for $(1+1)$-dimensional Minkowski spacetime.

diagram of Figure 4.7.1; compare Figure 4.2.1, p. 134. This diagram, which is not proper, fails to represent correctly the connectedness of $\mathscr{I}^+$ and $\mathscr{I}^-$ when $n>1$.

It will be seen in Section 4.7.8 below that yet another choice of π and of the set $(\mathscr{U},g)\subset\mathbb{R}^{1,n}$ leads to a third projection diagram for Minkowski spacetime.

A further example of non-uniqueness is provided by the projection diagrams for Taub-NUT metrics, discussed in Section 4.8.2.

These examples show that there is no uniqueness in the projection diagrams, and that various such diagrams might carry different information about the causal structure. It is clear that for spacetimes with intricate causal structure, some information will be lost when projecting to two dimensions. This raises the interesting question, whether there exists a notion of optimal projection diagram for specific spacetimes. In any case, the examples we give in what follows appear to depict the essential causal properties of the associated spacetime, except perhaps for the black ring diagrams of Section 4.7.8-4.7.9.

Non-trivial examples of metrics of the form (4.7.1) are provided by the Gowdy metrics on a torus [141]. These are vacuum $\mathrm{U}(1)\times\mathrm{U}(1)$-symmetric metrics which can globally be written in the form [57, 141]

$$g=e^f(-dt^2+d\theta^2)+|t|\left(e^P\left(dx^1+Q\,dx^2\right)^2+e^{-P}(dx^2)^2\right), \tag{4.7.3}$$

with $t\in(-\infty,0)$ and $(\theta,x^1,x^2)\in S^1\times S^1\times S^1$. Unwrapping θ from S^1 to $\mathbb{R}$ and projecting away the x^1 and x^2 coordinates, one obtains a projection diagram the image of which is the half-space $t<0$ in Minkowski spacetime. This can be further compactified as in Section 4.2.4, keeping in mind that the asymptotic behavior of the metric for large negative values of t [245] is not compatible with the existence of a smooth conformal completion of the full spacetime metric across past null infinity. Note that this projection diagram fails to represent properly the existence of Cauchy horizons for non-generic [246] Gowdy metrics.

Similarly, generic Gowdy metrics on $S^1\times S^2$, S^3, or $L(p,q)$ can be written in the form [57, 141]

$$g=e^f(-dt^2+d\theta^2)+R_0\sin(t)\sin(\theta)\left(e^P\left(dx^1+Q\,dx^2\right)^2+e^{-P}(dx^2)^2\right), \tag{4.7.4}$$

with $(t,\theta)\in(0,\pi)\times[0,\pi]$, leading to the Gowdy square as the projection diagram for the spacetime. (This is the diagram of Figure 4.7.13, p. 190, where the lower boundary corresponds to $t = 0$, the upper boundary corresponds to $t = \pi$, the left boundary corresponds to the axis of rotation $\theta = 0$, and the right boundary is the projection of the axis of rotation $\theta = \pi$. The diagonals, denoted as $y = y_h$ in Figure 4.7.13, correspond in the Gowdy case to the projection of the set where the gradient of the area $R = R_0 \sin(t)\sin(\theta)$ of the orbits of the isometry group $\mathrm{U}(1) \times \mathrm{U}(1)$ becomes null or vanishes, and do not have any further geometric significance. The lines with the arrows in Figure 4.7.13 are irrelevant for the Gowdy metrics, as the orbits of the isometry group of the spacetime metric, which are spacelike throughout the Gowdy square, have been projected away.)

Let us now pass to the construction of projection diagrams for families of metrics of interest which are not of the simple form (4.7.1).

4.7.3 The Kerr metrics

Consider the Kerr metric in Boyer-Lindquist coordinates,

$$\begin{aligned} g &= -\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 - \frac{2a\sin^2(\theta)\left(r^2+a^2-\Delta\right)}{\Sigma}dtd\varphi \\ &\quad + \frac{\sin^2(\theta)\left(\left(r^2+a^2\right)^2 - a^2\sin^2(\theta)\Delta\right)}{\Sigma}d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \,. \quad (4.7.5)\end{aligned}$$

Here

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 + a^2 - 2mr = (r-r_+)(r-r_-)\,, \quad (4.7.6)$$

for some real parameters a and m, with

$$r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}}\,, \quad \text{and we assume that } 0 < |a| \le m\,.$$

Recall that in the region $r \le 0$ there exists a non-empty domain on which the Killing vector ∂_φ becomes timelike:

$$\begin{aligned} \mathscr{V} &= \{g_{\varphi\varphi} < 0\} \\ &= \{r < 0\,,\ \cos(2\theta) < -\frac{a^4 + 2a^2mr + 3a^2r^2 + 2r^4}{a^2\Delta}\,, \\ &\qquad \Sigma \ne 0\,,\ \sin(\theta) \ne 0\} \qquad (4.7.7)\end{aligned}$$

(see (1.6.48)). Since the orbits of ∂_φ are periodic, this leads to causality violations, as described in detail in Section 1.6.3. But, as pointed out above, the existence of a projection diagram implies stable causality of the spacetime. It is thus clear that we will need to remove the region where ∂_φ is timelike to construct the diagram. It is, however, not clear whether simply removing $\mathscr{V}$ from $\mathscr{M}$ suffices. Now, in the construction below we will project-out the θ and φ coordinates, and we will see that removing those values of r which correspond to $\mathscr{V}$ suffices indeed.

We start by recalling (cf. (1.6.47), p. 78)

$$\begin{aligned} g_{\varphi\varphi} &= \sin^2(\theta)\left(\frac{2a^2mr\sin^2(\theta)}{a^2\cos^2(\theta)+r^2}+a^2+r^2\right) \\ &= \frac{\sin^2(\theta)\left(a^4+a^2\cos(2\theta)\Delta+a^2r(2m+3r)+2r^4\right)}{a^2\cos(2\theta)+a^2+2r^2}\,, \end{aligned} \tag{4.7.8}$$

where the first line makes clear the non-negativity of $g_{\varphi\varphi}$ for $r \geq 0$.

To fulfill the requirements of our definition, we will be projecting-out the θ and φ variables. We thus need to find a two-dimensional metric γ with the property that g-timelike vectors $X^t\partial_t + X^r\partial_r + X^\theta\partial_\theta + X^\varphi\partial_\varphi$ project to γ-timelike vectors $X^t\partial_t + X^r\partial_r$. For this, in the region where ∂_φ is spacelike (which thus includes $\{r > 0\}$) it turns out to be convenient to rewrite the $t-\varphi$ part of the metric as

$$\begin{aligned} &g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 \\ &\quad = g_{\varphi\varphi}\left(d\varphi + \frac{g_{t\varphi}}{g_{\varphi\varphi}}dt\right)^2 + \left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)dt^2\,, \end{aligned} \tag{4.7.9}$$

with

$$g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} = -\frac{2\Delta\Sigma}{a^4+a^2\Delta\cos(2\theta)+a^2r(2m+3r)+2r^4}\,.$$

For $r > 0$ and $\Delta > 0$ it holds that

$$\frac{\Delta\Sigma}{(a^2+r^2)^2} \leq |g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}| \leq \frac{\Delta\Sigma}{r\left(a^2(2m+r)+r^3\right)}\,, \tag{4.7.10}$$

with the infimum attained at $\theta \in \{0, \pi\}$ and maximum at $\theta = \pi/2$. One of the key facts for us is that $g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}$ has constant sign, and is in fact negative in this region.

In the region $r > 0$, $\Delta > 0$ consider any vector

$$X = X^t\partial_t + X^r\partial_r + X^\theta\partial_\theta + X^\varphi\partial_\varphi$$

which is causal for the metric g. Let $\Omega(r, \theta)$ be any strictly positive function. Since both $g_{\theta\theta}$ and the first term in (4.7.9) are positive, while the coefficient of dt^2 in (4.7.9) is negative, we have

$$\begin{aligned} 0 &\geq \Omega^2 g(X,X) = \Omega^2 g_{\mu\nu}X^\mu X^\nu \geq \Omega^2\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)(X^t)^2 + \Omega^2 g_{rr}(X^r)^2 \\ &\geq \inf_\theta\left(\Omega^2\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)(X^t)^2 + \Omega^2 g_{rr}(X^r)^2\right) \\ &\geq -\sup_\theta\,(\Omega^2|g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}|)(X^t)^2 + \inf_\theta\left(\Omega^2 g_{rr}\right)(X^r)^2\,. \end{aligned} \tag{4.7.11}$$

Thus, regardless of the choice of Ω, g-causality of X enforces a sign on the expression given in the last line of (4.7.11). We will therefore use this expression, with a suitable choice of Ω to define the desired projection metric γ It is simplest to choose Ω so that both extrema in (4.7.11) are attained at the same value of θ, say θ_*, while keeping those features of the coefficients which are essential for the problem at hand. It is convenient, but not essential, to have θ_* independent of r. We will therefore make the choice

$$\Omega^2 = \frac{r^2+a^2}{\Sigma} \,, \tag{4.7.12}$$

but other choices are possible, and might be more convenient for other purposes. Here the Σ factor has been included to get rid of the angular dependence in

$$\Omega^2 g_{rr} = \Omega^2 \frac{\Sigma}{\Delta}$$

while the numerator r^2+a^2 has been added to ensure that the metric coefficient γ_{rr} in (4.7.14) tends to one as r recedes to infinity. With this choice of Ω, (4.7.11) is equivalent to the statement that

$$\pi_*(X) := X^t\partial_t + X^r\partial_r \tag{4.7.13}$$

is a causal vector in the two-dimensional Lorentzian metric

$$\gamma := -\frac{\Delta(r^2+a^2)}{r\left(a^2(2m+r)+r^3\right)}\,dt^2 + \frac{(r^2+a^2)}{\Delta}dr^2\,. \tag{4.7.14}$$

Using the methods of Walker [274] reviewed in Section 4.3, in the region $r_+ < r < \infty$ the metric γ is conformal to a flat metric on the interior of a diamond, with the conformal factor extending smoothly across that part of its boundary at which $r \to r_+$ when $|a| < m$. This remains true when $|a| = m$ except at the leftmost corner i^0_L of Figure 4.7.1.

To make things clear, the map π of the definition of a projection diagram is the projection $(t, r, \theta, \varphi) \mapsto (t, r)$. The fact that g-causal curves are mapped to γ-causal curves follows from the construction of γ. In order to prove the lifting property, let $\sigma(s) = (t(s), r(s))$ be a γ-causal curve, then the curve

$$(t(s), r(s), \pi/2, \varphi(s))\,,$$

where $\varphi(s)$ satisfies

$$\frac{d\varphi}{ds} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\frac{dt}{ds}$$

is a g-causal curve which projects to σ.

For causal vectors in the region $r > 0$, $\Delta < 0$, we have instead

$$\begin{aligned} 0 \;&\geq\; \Omega^2 g(X,X) \geq \Omega^2\left(g_{tt} - \frac{g^2_{t\varphi}}{g_{\varphi\varphi}}\right)(X^t)^2 + \Omega^2 g_{rr}(X^r)^2 \\ &\geq\; \inf_\theta\left(\Omega^2\left|g_{tt} - \frac{g^2_{t\varphi}}{g_{\varphi\varphi}}\right|\right)(X^t)^2 - \sup_\theta\left(\Omega^2|g_{rr}|\right)(X^r)^2\,. \end{aligned} \tag{4.7.15}$$

Since the inequalities in (4.7.10) are reversed when $\Delta < 0$, choosing the same factor Ω one concludes again that $X^t\partial_t + X^r\partial_r$ is γ-causal in the metric (4.7.14)

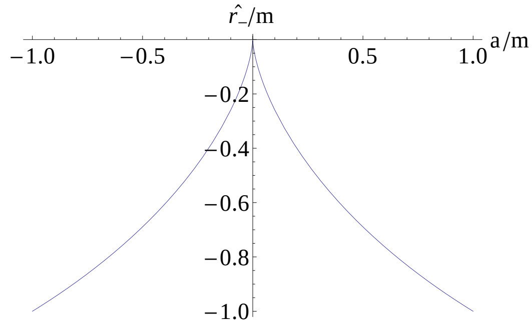

Figure 4.7.2: The radius of the "left boundary" $\hat{r}_-/m$ of the time-machine region as a function of a/m.

whenever it is in the metric g. Using again the results of Section 4.3, in the region $r_- < r < r_+$, such a metric is conformal to a a flat two-dimensional metric on the interior of a diamond, with the conformal factor extending smoothly across those parts of its boundary where $r \to r_+$ or $r \to r_-$.

When $|a| < m$ the metric coefficients in γ extend analytically from the $(r > r_+)$–range to the $(r_- < r < r_+)$–range. As described in Section 4.3.1, one can then smoothly glue together four diamonds as above to a single diamond on which $r_- < r < \infty$.

The singularity of γ at $r = 0$ reflects the fact that the metric g is singular at $\Sigma = 0$. This singularity persists even if $m = 0$, which might at first seem surprising since then there is no geometric singularity at $\Sigma = 0$ anymore [43]. However, this singularity of γ reflects the singularity of the associated coordinates on Minkowski spacetime (compare (1.6.3), p. 64), with the set $r = 0$ in the projection metric corresponding to a boundary of the projection diagram.

For $r < 0$ we have $\Delta > 0$, and the inequality (4.7.11) still applies in the region where ∂_φ is spacelike. Setting

$$\mathscr{U} := \mathscr{M} \setminus \overline{\mathscr{V}}\,,$$

where $\mathscr{V}$ is given by (4.7.7), throughout $\mathscr{U}$ we have

$$\frac{a^4 + 2a^2mr + 3a^2r^2 + 2r^4}{a^2\left(a^2 - 2mr + r^2\right)} > 1 \quad \Longleftrightarrow \quad r\left(a^2(2m+r) + r^3\right) > 0\,. \qquad (4.7.16)$$

Equivalently,

$$r < \hat{r}_- := \frac{\sqrt[3]{\sqrt{3}\sqrt{a^6 + 27a^4m^2} - 9a^2m}}{3^{2/3}} - \frac{a^2}{\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{a^6 + 27a^4m^2} - 9a^2m}} < 0\,, \qquad (4.7.17)$$

see Figure 4.7.2. In the region $r < \hat{r}_-$ the inequalities (4.7.10) hold again, and so the projected vector $\pi_*(X)$ as defined by (4.7.13) is causal, for g-causal X, in the metric γ given by (4.7.14). One concludes that the four-dimensional region $\{-\infty < r < r_-\}$ has the causal structure which projects to those diamonds of, e.g., Figure 4.7.3 with $\hat{r}_+ = 0$ which contain a shaded region. Those shaded

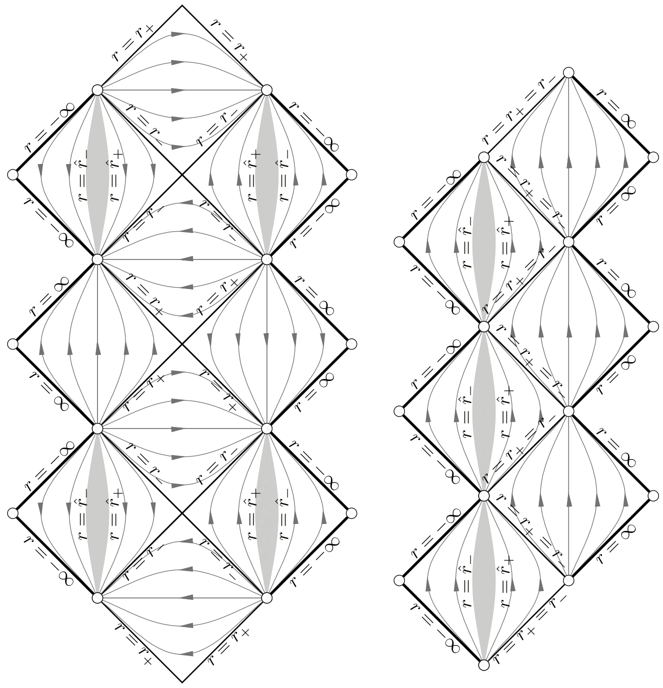

Figure 4.7.3: A projection diagram for the Kerr-Newman metrics with two distinct zeros of Δ (left diagram) and one double zero (right diagram); see Remark 4.7.2. In the Kerr case $Q = 0$ we have $\hat{r}_+ = 0$, with $\hat{r}_-$ given by (4.7.16).

regions, which correspond to the projection of both the singularity $r = 0$, $\theta = \pi/2$ and the time-machine region $\mathscr{V}$ of (4.7.7), belong to $\mathscr{W} = \pi(\mathscr{M})$ but *not* to $\pi(\mathscr{U})$. Causality within the shaded region is *not* represented in any useful way by a flat two-dimensional metric there, as causal curves can exit this region earlier, in Minkowskian time on the diagram, than they entered it. This results in causality violations throughout the enclosing diamond unless the shaded region is removed.

The projection diagrams for the usual maximal extensions of the Kerr-Newman metrics can be found in Figure 4.7.3.

REMARK 4.7.2 Some general remarks concerning projection diagrams for the Kerr family of metrics are in order. Anticipating, the remarks here apply also to projection diagrams of Kerr-Newman metrics, with or without a cosmological constant, to be discussed in the sections to follow.

The shaded regions in figures such as Figure 4.7.3 and others contain the singularity $\Sigma = 0$ and the time-machine set $\{g_{\varphi\varphi} < 0\}$, they belong to the set $\mathscr{W} = \pi(\mathscr{M})$ but do *not* belong to the set $\pi(\mathscr{U})$, on which causality properties of two-dimensional Minkowski spacetime reflect those of $\mathscr{U} \subset \mathscr{M}$. We emphasise that there are closed timelike curves through every point in the preimage under π of the entire diamonds containing the shaded areas; this is discussed in detail for the Kerr metric in Section 1.6.3, and applies *as is* to all metrics under consideration here. On the other hand, if the preimages of the shaded region are removed from $\mathscr{M}$, the causality relations in the resulting spacetimes are accurately represented by the diagrams, which are then proper.

The parameters $\hat{r}_\pm$ are determined by the mass and the charge parameters (see (4.7.54)), with $\hat{r}_+ = 0$ when the charge e vanishes, and $\hat{r}_+$ positive otherwise. The boundaries $r = \pm\infty$ correspond to smooth conformal boundaries at infinity, with causal character determined by Λ. The arrows indicate the spatial or timelike character of the orbits of the isometry group.

Maximal diagrams are obtained when continuing the diagrams shown in all allowed directions. It should be kept in mind that the resulting subsets of $\mathbb{R}^2$ are not simply connected in some cases, which implies that many alternative non-isometric maximal extensions of the spacetime can be obtained by taking various coverings of the planar diagram. One can also make use of the symmetries of the diagram to produce distinct quotients. □

Uniqueness of extensions

Let us denote by $(\mathscr{M}_{\rm Kerr}, g_{\rm Kerr})$ the spacetime with projection diagram visualised in Figure 4.7.3, continued indefinitely to the future and the past in the obvious way, including the preimages of the shaded regions except for the singular set $\{\Sigma = 0\}$. Note that $(\mathscr{M}_{\rm Kerr}, g_{\rm Kerr})$ is *not* simply connected because loops circling around $\{\Sigma = 0\}$ cannot be homotoped to a point. Let us denote by $(\widehat{\mathscr{M}}_{\rm Kerr}, \widehat{g}_{\rm Kerr})$ the universal covering space of $(\mathscr{M}_{\rm Kerr}, g_{\rm Kerr})$ with the pull-back metric.

The question then arises of the uniqueness of the extensions so obtained. To address this, we start by noting the following result of Carter [43] (compare [225, Theorem 4.3.1, p. 189]):

PROPOSITION 4.7.3 *The Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded on all maximally extended <u>incomplete</u> causal geodesics in $(\mathscr{M}_{\rm Kerr}, g_{\rm Kerr})$.*

Proposition 4.7.3 together with Corollary 1.4.7, p. 58, implies:

THEOREM 4.7.4 *Let $(\mathscr{M}, g)$ denote the region $\{r > r_+\}$ of a Kerr metric with $|a| \le m$. Then $(\widehat{\mathscr{M}}_{\rm Kerr}, \widehat{g}_{\rm Kerr})$ is the unique simply connected analytic extension of $(\mathscr{M}, g)$ such that all maximally extended causal geodesics along which $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded are complete.* □

Theorem 4.7.4 makes it clear in which sense $(\widehat{\mathscr{M}}_{\rm Kerr}, \widehat{g}_{\rm Kerr})$ is unique. However, the extension $(\mathscr{M}_{\rm Kerr}, g_{\rm Kerr})$ appears to be more economical, if not more natural. It would be of interest to find a natural condition which singles it out.

Conformal diagrams for a class of two-dimensional submanifolds of Kerr spacetime

One can find e.g. in [45, 147] conformal diagrams for the symmetry axes in the maximally extended Kerr spacetime. These diagrams are identical with those of

Figure 4.7.3, except for the absence of shading. (The authors of [45, 147] seem to indicate that the subset $r = 0$ plays a special role in their diagrams, which is not the case as the singularity $r = \cos\theta = 0$ does not intersect the symmetry axes.) Now, the symmetry axes are totally geodesic submanifolds, being the collection of fixed points of the isometry group generated by the rotational Killing vector field. They can be thought of as the submanifolds $\theta = 0$ and $\theta = \pi$ (with the remaining angular coordinate irrelevant then) of the extended Kerr spacetime. As such, another totally geodesic two-dimensional submanifold in Kerr is the equatorial plane $\theta = \pi/2$, which is the set of fixed points of the isometry $\theta \mapsto \pi - \theta$. This leads one to enquire about the global structure of this submanifold or, more generally, of various families of two-dimensional submanifolds on which θ is kept fixed. The discussion that follows illustrates clearly the distinction between projection diagrams, in which one projects-out the θ and φ variables, and conformal diagrams for submanifolds where θ, and φ or the angular variable $\tilde\varphi$ of (4.7.20) below, are fixed.

An obvious family of two-dimensional Lorentzian submanifolds to consider is that of submanifolds, which we denote as $N_{\theta,\varphi}$, which are obtained by keeping θ and φ fixed. The metric, say $g(\theta)$, induced by the Kerr metric on $N_{\theta,\varphi}$ reads

$$g(\theta) \quad = \quad -\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 =: -F_1(r)dt^2 + \frac{dr^2}{F_2(r)} \,. \tag{4.7.18}$$

For $m^2 - a^2\cos^2(\theta) > 0$ the function F_1 has two first-order zeros at the intersection of $N_{\theta,\varphi}$ with the boundary of the ergoregion $\{g(\partial_t, \partial_t) > 0\}$:

$$r_{\theta,\pm} = m \pm \sqrt{m^2 - a^2\cos^2(\theta)}\,. \tag{4.7.19}$$

The key point is that these zeros are distinct from those of F_2 if $\cos^2\theta \neq 1$, which we assume in the remainder of this section. Since $r_{\theta,+}$ is larger than the largest zero of F_2, the metric $g(\theta)$ is a priori only defined for $r > r_{\theta,+}$. One checks that its Ricci scalar diverges as $(r - r_{\theta,+})^{-2}$ when $r_{\theta,+}$ is approached, therefore those submanifolds do not extend smoothly across the ergosphere, and will thus be of no further interest to us.

We consider, next, the two-dimensional submanifolds, say $\tilde N_{\theta,\tilde\varphi}$, of the Kerr spacetime obtained by keeping θ and $\tilde\varphi$ fixed, where $\tilde\varphi$ is a new angular coordinate defined as

$$d\tilde\varphi = d\varphi + \frac{a}{\Delta}\,dr\,. \tag{4.7.20}$$

Using further the coordinate v defined as

$$dv \quad = \quad dt + \frac{(a^2 + r^2)}{\Delta}dr\,, \tag{4.7.21}$$

the metric, say $\tilde g(\theta)$, induced on $\tilde N_{\theta,\tilde\varphi}$ takes the form

$$\begin{aligned}\tilde g(\theta) \quad &= \quad -\frac{\tilde F(r)}{\Sigma}dv^2 + 2dvdr \\ &= \quad -\frac{\tilde F(r)}{\Sigma}dv\left(dv - 2\frac{\Sigma}{\tilde F(r)}dr\right)\,,\end{aligned} \tag{4.7.22}$$

where $\tilde F(r) := r^2 + a^2\cos^2(\theta) - 2mr$. The zeros of $\tilde F(r)$ are again given by (4.7.19). Setting

$$du = dv - 2\frac{\Sigma}{\tilde F(r)}dr \tag{4.7.23}$$

brings (4.7.22) to the form

$$\tilde g(\theta) = -\frac{\tilde F(r)}{\Sigma}dvdu\,.$$

The usual Kruskal-Szekeres type of analysis applies to this metric, leading to a conformal diagram as in the left Figure 4.7.3 *with no shadings*, and with $r_\pm$ there replaced by $r_{\theta,\pm}$, *as long as $\tilde F$ has two distinct zeros.*

Several comments are in order:

First, the event horizons *within $\tilde N_{\theta,\tilde\varphi}$ do not* coincide with the intersection of the event horizons of the Kerr spacetime with $\tilde N_{\theta,\tilde\varphi}$. This is not difficult to understand by noting that the class of causal curves that lie within $\tilde N_{\theta,\tilde\varphi}$ is smaller than the class of causal curves in spacetime, and there is therefore no a priori reason to expect that the associated horizons will be the same. In fact, is should be clear that the event horizons within $\tilde N_{\theta,\tilde\varphi}$ should be located on the boundary of the ergoregion, since in two spacetime dimensions the boundary of an ergoregion is necessarily a null hypersurface. This illustrates the fact that conformal diagrams for submanifolds might fail to represent correctly the location of horizons. The reason that the conformal diagrams for the symmetry axes correctly reflect the global structure of the spacetime is an accident related to the fact that the ergosphere touches the event horizon there.

This last issue acquires a dramatic dimension for extreme Kerr black holes, for which $|a| = m$, where for $\theta \in (0,\pi)$ the global structure of maximally extended $\tilde N_{\theta,\tilde\varphi}$'s is represented by an unshaded version of the left Figure 4.7.3, while the conformal diagrams for the axisymmetry axes are given by the unshaded version of the right Figure 4.7.3.

Next, another dramatic change arises in the global structure of the $\tilde N_{\theta,\tilde\varphi}$'s with $\theta = \pi/2$. Indeed, in this case we have $r_{\theta,+} = 2m$, as in Schwarzschild spacetime, and $r_{\theta,-} = 0$, *regardless of whether the metric is underspinning, extreme, or overspinning.* Since $r_{\theta,-}$ coincides now with the location of the singularity, $\tilde N_{\theta,\tilde\varphi}$ acquires two connected components, one where $r > 0$ and a second one with $r < 0$. The conformal diagram of the first one is identical to that of the Schwarzschild spacetime with positive mass, while the second is identical to that of Schwarzschild with negative mass, see Figure 4.7.4. We thus obtain the unexpected conclusion, that *the singularity $r = \cos(\theta) = 0$ has a spacelike character when approached with positive r within the equatorial plane, and a timelike one when approached with negative r within that plane.* This is rather obvious in retrospect, since the metric induced by Kerr on $\tilde N_{\pi/2,\tilde\varphi}$ coincides, when $m > 0$, with the one induced by the Schwarzschild metric with positive mass in the region $r > 0$ and with the Schwarzschild metric with negative mass $-m$ in the region $r < 0$.

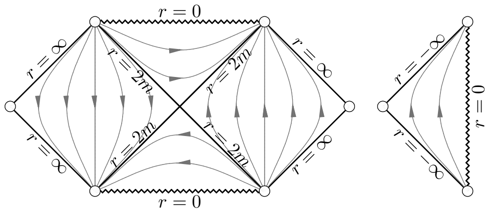

Figure 4.7.4: The conformal diagram for a maximal analytic extension of the metric induced by the Kerr metric, with arbitrary $a \in \mathbb{R}$, on the submanifolds of constant angle $\tilde{\varphi}$ within the equatorial plane $\theta = \pi/2$, with $r > 0$ (left) and $r < 0$ (right).

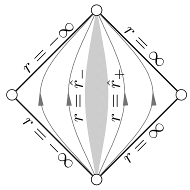

Figure 4.7.5: A projection diagram for overspinning Kerr-Newman spacetimes.

Note finally that, surprisingly enough, even for overspinning Kerr metrics there will be a range of angles θ near $\pi/2$ so that $\tilde{F}$ will have two distinct first-order zeros. This implies that, for such θ, the global structure of maximally extended $\tilde{N}_{\theta,\tilde{\varphi}}$'s will be similar to that of the corresponding submanifolds of the underspinning Kerr solutions. This should be compared with the projection diagram for overspinning Kerr spacetimes in Figure 4.7.5.

The orbit-space metric on $\mathscr{M}/\mathrm{U}(1)$

Let h denote the tensor field obtained by quotienting-out in the Kerr metric g the $\eta := \partial_\varphi$ direction,

$$h(X,Y) = g(X,Y) - \frac{g(X,\eta)g(Y,\eta)}{g(\eta,\eta)}\,. \tag{4.7.24}$$

(Compare Section 1.6.6, where the whole group $\mathbb{R} \times \mathrm{U}(1)$ has been quotiented-out instead.) The tensor field h projects to the natural quotient metric on the manifold part of $\mathscr{M}/\mathrm{U}(1)$. In the region where η is spacelike, the quotient space $\mathscr{M}/\mathrm{U}(1)$ has the natural structure of a manifold with boundary, where the boundary is the image, under the quotient map, of the axis of rotation

$$\mathscr{A} := \{\eta = 0\}\,.$$

Using t, r, θ as coordinates on the quotient space we find a diagonal metric

$$h = h_{tt}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,, \tag{4.7.25}$$

where
$$h_{tt} = g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\,,$$
as in (4.7.9). Thus, the metric γ of (4.7.14) is directly constructed out of the (t,r)–part of the quotient-space metric h. However, the analogy is probably misleading as there does not seem to be any direct correspondence between the quotient space $\mathscr{M}/\mathrm{U}(1)$ and the natural manifold as constructed in Section 4.7.3 using the metric γ.

We note that a Penrose diagram for the quotient-space metric has been constructed in [138]. The Penrose-Carter conformal diagram of Section 4.6 of [138] coincides with a projection diagram for the BMPV metric, but our interpretation of this diagram differs.

4.7.4 The Kerr-Newman metrics

The analysis of the Kerr-Newman metrics is essentially identical to that of the Kerr metric: The metric takes the same general form (4.7.5), except that now
$$\Delta = r^2 + a^2 + e^2 - 2mr =: (r-r_+)(r-r_-)\,,$$
and we assume that $e^2 + a^2 \le m$ so that the roots are real. We have
$$g_{\varphi\varphi} = \frac{\sin^2(\theta)\left(\left(r^2+a^2\right)^2 - a^2\Delta\sin^2(\theta)\right)}{\Sigma}\,, \tag{4.7.26}$$
$$g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} = -\frac{\Delta\Sigma}{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2(\theta)}\,, \tag{4.7.27}$$
and note that the sign of the denominator in (4.7.27) coincides with the sign of $g_{\varphi\varphi}$. Hence
$$\mathrm{sign}(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}) = -\mathrm{sign}(\Delta)\mathrm{sign}(g_{\varphi\varphi})\,.$$
For $g_{\varphi\varphi} > 0$, which is the main region of interest, we conclude that the minimum of $(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}})\Sigma^{-1}\Delta^{-1}$ is assumed at $\theta = \frac{\pi}{2}$ and the maximum at $\theta = 0, \pi$, so for all r for which $g_{\varphi\varphi} > 0$ we have
$$-\frac{\Delta\Sigma}{\left(r^2+a^2\right)^2 - a^2\Delta} \le g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \le -\frac{\Delta\Sigma}{\left(r^2+a^2\right)^2}\,. \tag{4.7.28}$$
Choosing the conformal factor as
$$\Omega^2 = \frac{r^2+a^2}{\Sigma}$$
we obtain, for g-causal vectors X,
$$\begin{aligned} 0 \ &\ge\ \Omega^2 g(X,X) = \Omega^2 g_{\mu\nu}X^\mu X^\nu \ge \Omega^2\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)(X^t)^2 + \Omega^2 g_{rr}(X^r)^2 \\ &\ge\ -\frac{\Delta\left(r^2+a^2\right)}{\left(r^2+a^2\right)^2 - a^2\Delta}(X^t)^2 + \frac{\left(r^2+a^2\right)}{\Delta}(X^r)^2\,. \end{aligned} \tag{4.7.29}$$

This leads to the following projection metric

$$\begin{aligned}\gamma &:= -\frac{\Delta\left(r^2+a^2\right)}{\left(r^2+a^2\right)^2-a^2\Delta}\,dt^2+\frac{\left(r^2+a^2\right)}{\Delta}dr^2\\ &= -\frac{\Delta\left(r^2+a^2\right)}{a^2\left(r(2m+r)-e^2\right)+r^4}\,dt^2+\frac{\left(r^2+a^2\right)}{\Delta}dr^2\,, \qquad (4.7.30)\end{aligned}$$

which is Lorentzian if and only if r is such that $g_{\varphi\varphi}>0$ for all $\theta\in[0,\pi]$.

Now, it follows from (4.7.26) that $g_{\varphi\varphi}$ will have the *wrong* sign if

$$0 \;>\; \left(r^2+a^2\right)^2-a^2\Delta\sin^2(\theta)\,. \qquad (4.7.31)$$

This does not happen when $\Delta\le 0$, and hence in a neighborhood of both horizons. On the other hand, for $\Delta>0$, a necessary condition for (4.7.31) is

$$0 \;>\; \left(r^2+a^2\right)^2-a^2\Delta = r^4+r^2a^2+2\,mra^2-a^2e^2 =: f(r)\,. \qquad (4.7.32)$$

The second derivative of f is strictly positive, hence f' has exactly one real zero. Note that f is strictly smaller than the corresponding function for the Kerr metric, where $e=0$, thus the interval where f is strictly negative encloses the corresponding interval for Kerr. We conclude that f is negative on an interval $(\hat r_-,\hat r_+)$, with $\hat r_- <0<\hat r_+<r_-$.

The corresponding projection diagrams are identical to those of the Kerr spacetime, see Figure 4.7.3, with the minor modification that the region to be excised from the diagram is $\{r\in(\hat r_-,\hat r_+)\}$, with now $\hat r_+>0$, while we had $\hat r_+=0$ in the uncharged case.

4.7.5 The Kerr - de Sitter metrics

The Kerr - de Sitter metric in Boyer-Lindquist-like coordinates reads [44, 106]

$$\begin{aligned}g &= \frac{\Sigma}{\Delta_r}dr^2+\frac{\Sigma}{\Delta_\theta}d\theta^2+\frac{\sin^2(\theta)}{\Xi^2\Sigma}\Delta_\theta\left(adt-(r^2+a^2)d\varphi\right)^2\\ &\quad -\frac{1}{\Xi^2\Sigma}\Delta_r\left(dt-a\sin^2(\theta)\,d\varphi\right)^2\,, \qquad (4.7.33)\end{aligned}$$

where

$$\Sigma=r^2+a^2\cos^2(\theta)\,,\qquad \Delta_r=(r^2+a^2)\left(1-\frac{\Lambda}{3}r^2\right)-2\mu\Xi r\,, \qquad (4.7.34)$$

and

$$\Delta_\theta=1+\frac{\Lambda}{3}a^2\cos^2(\theta)\,,\qquad \Xi=1+\frac{\Lambda}{3}a^2\,, \qquad (4.7.35)$$

for some real parameters a and μ, where Λ is the cosmological constant. In this section we assume

$$\Lambda>0 \text{ and } a\neq 0.$$

By a redefinition $\varphi \mapsto -\varphi$ we can always achieve $a > 0$, similarly changing r to $-r$ if necessary we can assume that $\mu \geq 0$. The case $\mu = 0$ leads to the de Sitter metric in unusual coordinates (see, e.g., [4, Equation (17)]). The inequalities

$$a > 0 \text{ and } \mu > 0$$

will be assumed from now on.

The Lorentzian character of the metric should be clear from (4.7.33); alternatively, one can calculate the determinant of g:

$$\det(g) = -\frac{\Sigma^2}{\Xi^4} \sin^2\theta \,. \tag{4.7.36}$$

We have

$$g^{tt} = \frac{g_{rr}\, g_{\theta\theta}\, g_{\varphi\varphi}}{\det(g)} = -\frac{\Xi^4}{\Delta_\theta} \times \frac{1}{\Delta_r} \times \frac{g_{\varphi\varphi}}{\sin^2\theta} \,, \tag{4.7.37}$$

which shows that either t or its negative is a time function whenever Δ_r and $g_{\varphi\varphi}/\sin^2\theta$ are strictly positive. (Incidentally, chronology is violated on the set where $g_{\varphi\varphi} < 0$, we will return to this shortly.) One also has

$$g^{rr} = \frac{\Delta_r}{\Sigma} \,, \tag{4.7.38}$$

which shows that r or its negative is a time function in the region where $\Delta_r < 0$.

The character of the principal orbits of the isometry group $\mathbb{R} \times U(1)$ is determined by the sign of the determinant

$$\det \begin{pmatrix} g_{tt} & g_{\varphi t} \\ g_{\varphi t} & g_{\varphi\varphi} \end{pmatrix} = -\frac{\Delta_r \Delta_\theta}{\Xi^4} \sin^2\theta \,. \tag{4.7.39}$$

Therefore, for $\sin(\theta) \neq 0$ the orbits are two-dimensional, timelike in the regions where $\Delta_r > 0$, spacelike where $\Delta_r < 0$, and null where $\Delta_r = 0$ once the spacetime has been appropriately extended to include the last set.

When $\mu \neq 0$ the set $\{\Sigma = 0\}$ corresponds to a geometric singularity in the metric. To see this, note that

$$g(\partial_t, \partial_t) = \frac{a^2 \sin^2\theta \Delta_\theta - \Delta_r}{\Sigma\, \Xi^2} = 2\, \frac{\mu\, r}{\Sigma\, \Xi} + O(1) \,, \tag{4.7.40}$$

where $O(1)$ denotes a function which is bounded near $\Sigma = 0$. It follows that for $\mu \neq 0$ the norm of the Killing vector ∂_t blows up as the set $\{\Sigma = 0\}$ is approached along the plane $\cos(\theta) = 0$, which would be impossible if the metric could be continued across this set in a C^2 manner.

The function Δ_r has exactly *two distinct first-order real* zeros, one of them strictly negative and the other strictly positive, when

$$\mu^2 > \frac{2}{3^5 \Xi^2 \Lambda} \left(3 - a^2\Lambda\right)^3 \,. \tag{4.7.41}$$

It has *at least two, and up to four, possibly but not necessarily distinct,* real roots when

$$a^2\Lambda \leq 3 \,, \quad \mu^2 \leq \frac{2}{3^5 \Xi^2 \Lambda} \left(3 - a^2\Lambda\right)^3 \,. \tag{4.7.42}$$

Under the current assumptions the smallest root, say r_1, is always simple and strictly negative, the remaining ones are strictly positive. We can thus order the roots as

$$r_1 < 0 < r_2 \le r_3 \le r_4 \,, \tag{4.7.43}$$

when there are four real ones, and we set $r_3 \equiv r_4 := r_2$ when there are only two real roots $r_1 < r_2$. The function Δ_r is strictly positive for $r \in (r_1, r_2)$, and for $r \in (r_3, r_4)$ whenever the last interval is not empty; Δ_r is negative or vanishing otherwise.

It holds that

$$\begin{aligned} g_{\varphi\varphi} &= \frac{\sin^2(\theta)\left(\Delta_\theta\left(r^2+a^2\right)^2 - a^2\Delta_r\sin^2(\theta)\right)}{\Xi^2\Sigma} && (4.7.44)\\ &= \frac{\sin^2(\theta)}{\Xi}\left(\frac{2a^2\mu r\sin^2(\theta)}{a^2\cos^2(\theta)+r^2} + a^2 + r^2\right). && (4.7.45)\end{aligned}$$

The second line is manifestly non-negative for $r \ge 0$, and positive there away from the axis $\sin(\theta) = 0$. The first line is manifestly non-negative for $\Delta_r \le 0$, and hence also in a neighborhood of this set.

Next

$$\begin{aligned} g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} &= -\frac{\Delta_\theta\Delta_r\Sigma}{\Xi^2\left(\Delta_\theta\left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2(\theta)\right)} \\ &= -\frac{\Delta_\theta\Delta_r\Sigma}{\Xi^2\left(A(r)+B(r)\cos(2\theta)\right)}\,, && (4.7.46)\end{aligned}$$

with

$$\begin{aligned} A(r) &= \frac{\Xi}{2}\left(a^4 + 3a^2r^2 + 2r^4 + 2a^2\mu r\right), && (4.7.47)\\ B(r) &= \frac{a^2}{2}\Xi\left(a^2 + r^2 - 2\mu r\right). && (4.7.48)\end{aligned}$$

We have

$$\begin{aligned} A(r)+B(r) &= \Xi\left(a^2+r^2\right)^2,\\ A(r)-B(r) &= r^2\Xi\left(a^2+r^2+2\frac{a^2\mu}{r}\right), && (4.7.49)\end{aligned}$$

which confirms that for $r > 0$, or for large negative r, we have $A > |B| > 0$, as needed for $g_{\varphi\varphi} \ge 0$. The function

$$f(r,\theta) := \frac{(A(r)+B(r)\cos(2\theta))}{\Delta_\theta} \equiv \frac{(A(r)+B(r)\cos(2\theta))}{1+\frac{\Lambda}{3}a^2\cos^2(\theta)}$$

satisfies

$$\frac{\partial f}{\partial\theta} = -\frac{a^2\Xi}{\Delta_\theta^2}\Delta_r\sin(2\theta)\,, \tag{4.7.50}$$

which has the same sign as $-\Delta_r \sin(2\theta)$. In any case, its extrema are achieved at $\theta = 0$, $\pi/2$ and π. Accordingly, this is where the extrema of the right-hand side of (4.7.46) are achieved as well. In particular, for $\Delta_r > 0$, we find

$$\frac{\Delta_r \Sigma}{(a^2+r^2)^2} \leq \Xi^2 \left| g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right| \leq \frac{\Sigma \Delta_r}{\Xi r \left(a^2(2\mu + r) + r^3\right)}, \tag{4.7.51}$$

with the minimum attained at $\theta = 0$ and the maximum attained at $\theta = \pi/2$.

To obtain the projection diagram, we can now repeat word for word the analysis carried out for the Kerr metrics on the set $\{g_{\varphi\varphi} > 0\}$. Choosing a conformal factor Ω^2 equal to

$$\Omega^2 = \frac{r^2 + a^2}{\Sigma}, \tag{4.7.52}$$

one is led to a projection metric

$$\gamma := -\frac{(r^2+a^2)\Delta_r}{\Xi^3 r \left(a^2(2\mu + r) + r^3\right)} dt^2 + \frac{r^2+a^2}{\Delta_r} dr^2 . \tag{4.7.53}$$

It remains to understand the set

$$\mathscr{V} \quad := \quad \{g_{\varphi\varphi} < 0\}$$

where $g_{\varphi\varphi}$ is negative. To avoid repetitiveness, we will do it simultaneously both for the charged and the uncharged case, where (4.7.44) still applies (but not (4.7.45) for $e \neq 0$) with Δ_r given by (4.7.54); the Kerr - de Sitter case is obtained by setting $e = 0$ in what follows. A calculation shows that $g_{\varphi\varphi}$ is the product of a non-negative function with

$$\chi := 2\,a^2 \mu\, r - a^2 e^2 + r^2 a^2 + r^4 + \left(r^2 a^2 - 2\,a^2 \mu\, r + a^2 e^2 + a^4\right) \cos^2(\theta) .$$

This is clearly strictly positive for all r and all $\theta \neq \pi/2$ when $\mu = e = 0$, which shows that $\mathscr{V} = \emptyset$ in this case.

Next, the function χ is sandwiched between the two following functions of r, obtained by setting $\cos(\theta) = 0$ or $\cos^2(\theta) = 1$ in χ:

$$\begin{aligned} \chi_0 &:= r^4 + r^2 a^2 + 2\,a^2 \mu\, r - a^2 e^2 , \\ \chi_1 &:= \left(r^2 + a^2\right)^2 . \end{aligned}$$

Hence, χ is strictly positive for all r when $\cos^2(\theta) = 1$. Next, for $\mu > 0$ the function χ_0 is negative for negative r near zero. Further, χ_0 is convex. We conclude that, for $\mu > 0$, the set on which χ_0 is non-positive is a non-empty interval $[\hat r_-, \hat r_+]$ containing the origin. We have already seen that $g_{\varphi\varphi}$ is non-negative wherever $\Delta_r \leq 0$, and since $r_2 > 0$ we must have

$$r_1 < \hat r_- \leq \hat r_+ < r_2 .$$

In fact, when $e = 0$ the value of $\hat r_-$ is given by (4.7.17) with m there replaced by μ, with $\hat r_- = 0$ if and only if $\mu = 0$.

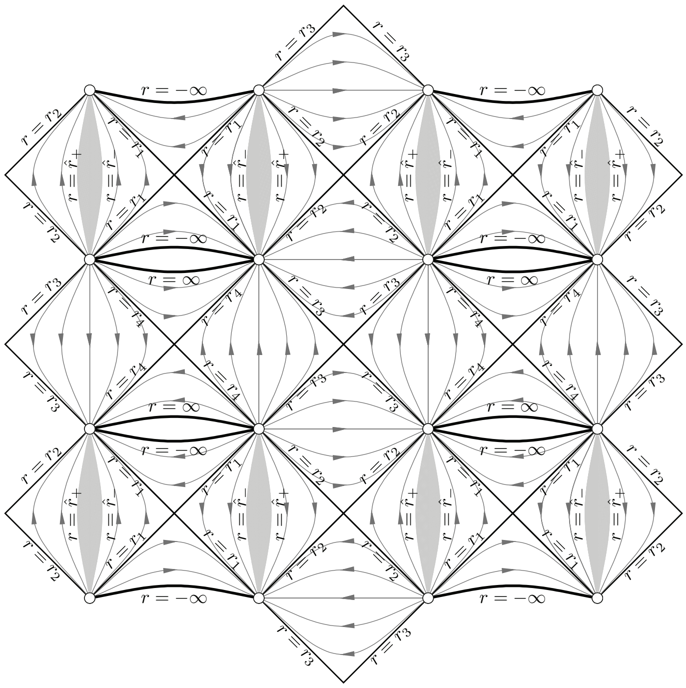

Figure 4.7.6: A projection diagram for the Kerr-Newman - de Sitter metric with four distinct zeros of Δ_r; see Remark 4.7.2.

We conclude that if $\mu = e = 0$ the time-machine set is empty, while if $|\mu| + e^2 > 0$ there are always causality violations "produced" in the non-empty region $\{\hat{r}_- \le r \le \hat{r}_+\}$.

The projection diagrams for the Kerr-Newman - de Sitter family of metrics depend upon the number of zeros of Δ_r, and their nature, and can be found in Figures 4.7.6-4.7.9.

4.7.6 The Kerr-Newman - de Sitter metrics

In the standard Boyer–Lindquist coordinates the Kerr-Newman - de Sitter metric takes the form (4.7.33) [44, 259],[3] with all the functions as in (4.7.34)-(4.7.35) except for Δ_r, which instead takes the form

$$\Delta_r = \left(1 - \tfrac{1}{3}\Lambda r^2\right)(r^2 + a^2) - 2\Xi\mu r + \Xi e^2\,, \qquad (4.7.54)$$

where $\sqrt{\Xi}e$ is the electric charge of the spacetime. In this section we assume

$$\Lambda > 0\,, \quad \mu \ge 0\,, \quad a > 0\,, \quad e \ne 0\,.$$

The calculations of the previous section, and the analysis of zeros of Δ_r,

[3]The transition from the formulae in [44] to (4.7.33) is explained in [45, p. 102].

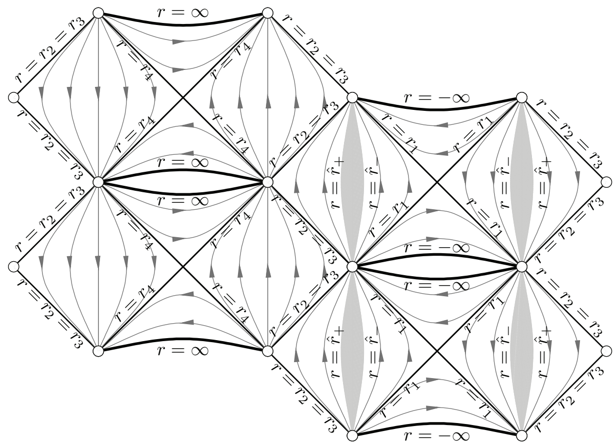

Figure 4.7.7: A projection diagram for the Kerr-Newman - de Sitter metrics with three distinct zeros of Δ_r, $r_1<0<r_2=r_3<r_4$; see Remark 4.7.2.

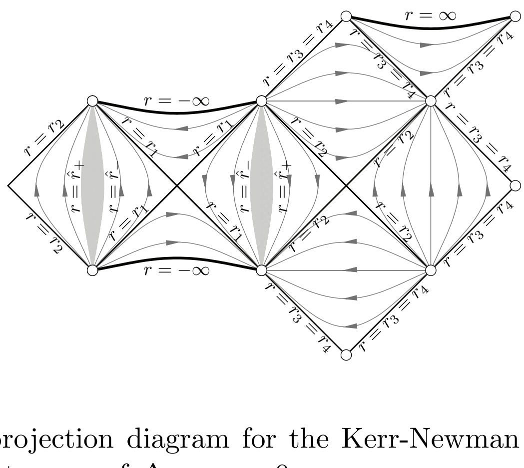

Figure 4.7.8: A projection diagram for the Kerr-Newman - de Sitter metrics with three distinct zeros of Δ_r, $r_1<0<r_2<r_3=r_4$; see Remark 4.7.2. Note that one cannot continue the diagram simultaneously across all boundaries $r=r_3$ on $\mathbb{R}^2$, but this can be done on an appropriate Riemann surface.

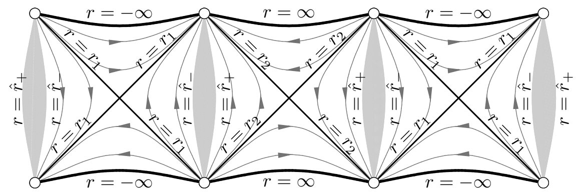

Figure 4.7.9: A projection diagram for the Kerr-Newman - de Sitter metrics with two distinct first-order zeros of Δ_r, $r_1 < 0 < r_2$ and $\mu > 0$; see Remark 4.7.2. The diagram for a first-order zero at r_1 and third-order zero at $r_2 = r_3 = r_4$ would be identical except for the bifurcation surface of the bifurcate Killing horizon at the intersection of the lines $r = r_2$, which does not exist in the third-order case and has therefore to be removed from the diagram.

remain identical except for the following equations: First,

$$g_{\varphi\varphi} = \frac{\sin^2(\theta)}{\Xi}\left(\frac{a^2(2\mu r - e^2)\sin^2(\theta)}{a^2\cos^2(\theta)+r^2} + a^2 + r^2\right) , \tag{4.7.55}$$

the sign of which requires further analysis, we will return to this shortly. Next, we still have

$$\begin{aligned} g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} &= -\frac{\Delta_\theta\Delta_r\Sigma}{\Xi^2\left(\Delta_\theta\left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2(\theta)\right)} \\ &= -\frac{\Delta_\theta\Delta_r\Sigma}{\Xi^2\left(A(r)+B(r)\cos(2\theta)\right)} , \end{aligned} \tag{4.7.56}$$

but now

$$A(r) = \frac{\Xi}{2}\left(a^4 + 3a^2r^2 + 2r^4 + 2a^2\mu r - a^2e^2\right) , \tag{4.7.57}$$

$$B(r) = \frac{a^2}{2}\Xi\left(a^2 + r^2 - 2\mu r + e^2\right) , \tag{4.7.58}$$

with

$$\begin{aligned} A(r)+B(r) &= \Xi\left(a^2+r^2\right)^2 , \\ A(r)-B(r) &= r^2\Xi\left(a^2 + r^2 + 2\frac{a^2\mu}{r} - \frac{a^2e^2}{r^2}\right) . \end{aligned} \tag{4.7.59}$$

Equation (4.7.50) remains unchanged, and for $\Delta_r > 0$, we find

$$\frac{\Delta_r\Sigma}{\left(a^2+r^2\right)^2} \le \Xi^2\left|g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right| \le \frac{\Sigma\Delta_r}{\Xi\left(a^2(2\mu r - e^2 + r^2) + r^4\right)} , \tag{4.7.60}$$

with the minimum attained at $\theta = 0$ and the maximum attained at $\theta = \pi/2$. This leads to the projection metric

$$\gamma := -\frac{\Delta_r}{\Xi^3\left(a^2(2\mu r - e^2 + r^2) + r^4\right)}\, dt^2 + \frac{1}{\Delta_r}dr^2 . \tag{4.7.61}$$

We recall that the analysis of the time-machine set $\{g_{\varphi\varphi} < 0\}$ has already been carried out at the end of Section 4.7.5, where it was shown that for $e \neq 0$ causality violations always exist, and arise from the non-empty region $\{\hat{r}_- \leq r \leq \hat{r}_+\}$.

The projection diagrams for the Kerr-Newman - de Sitter family of metrics can be found in Figures 4.7.6-4.7.9.

4.7.7 The Kerr-Newman - anti de Sitter metrics

We consider the metric (4.7.33)-(4.7.35), with however Δ_r given by (4.7.54), assuming that

$$a^2 + e^2 > 0\,, \quad \Lambda < 0\,.$$

While the local calculations carried out in Section 4.7.5 remain unchanged, one needs to reexamine the occurrence of zeros of Δ_r.

We start by noting that the requirement that $\Xi \neq 0$ imposes

$$1 + \frac{\Lambda}{3}a^2 \neq 0\,.$$

Next, a negative Ξ would lead to a function Δ_θ which changes sign. By inspection, one finds that the signature changes from $(-+++)$ to $(+---)$ across these zeros, which implies nonexistence of a coordinate system in which the metric could be smoothly continued there.[4] From now on we thus assume that

$$\Xi \equiv 1 + \frac{\Lambda}{3}a^2 > 0\,. \tag{4.7.62}$$

As such, those metrics for which Δ_r has no zeros are nakedly singular whenever

$$e^2 + |\mu| > 0\,. \tag{4.7.63}$$

This can be easily seen from the following formula for g_{tt} on the equatorial plane:

$$g_{tt} = \frac{1}{3\Xi^2 r^2}(-3\,\Xi\, e^2 + 6\,\Xi\,\mu\, r + \left(\Lambda\, a^2 - 3\right) r^2 + \Lambda\, r^4)\,. \tag{4.7.64}$$

So, under (4.7.63) the norm of the Killing vector ∂_t is unbounded and the metric cannot be C^2-continued across $\{\Sigma = 0\}$ by usual arguments.

Turning our attention, first, to the region where $r > 0$, the occurrence of zeros of Δ_r requires that

$$\mu \geq \mu_c(a, e, \Lambda) > 0\,.$$

Hence, there is a strictly positive threshold for the mass of a black hole at given a and e. The solution with $\mu = \mu_c$ has the property that Δ_r and its r-derivative have a joint zero, and can thus be found by equating to zero the resultant of these two polynomials in r. An explicit formula for $m_c = \Xi\mu_c$

[4] We, and Kayll Lake (private communication), calculated several curvature invariants for the overspinning metrics and found no singularity at $\Delta_\theta = 0$. The origin of this surprising fact is not clear to us.

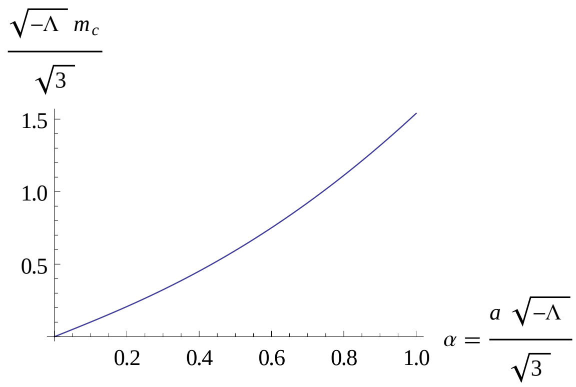

Figure 4.7.10: The critical mass parameter $m_c\sqrt{|\Lambda/3|} = \Xi\mu_c\sqrt{|3/\Lambda|}$ as a function of $|a|\sqrt{|\Lambda/3|}$ when $q = 0$.

can be given, which takes a relatively simple form when expressed in terms of suitably renormalised parameters. We set

$$\begin{aligned} \alpha = \sqrt{\frac{|\Lambda|}{3}}a \qquad &\Longleftrightarrow \qquad a = \alpha\sqrt{\frac{3}{|\Lambda|}}\,, \\ \gamma = 9\frac{\alpha^2 + \frac{|\Lambda|}{3}q^2}{(1+\alpha^2)^2} \qquad &\Longleftrightarrow \qquad q^2 := \Xi e^2 = \frac{3}{|\Lambda|}\left(\left(\frac{1+\alpha^2}{3}\right)^2\gamma - \alpha^2\right)\,, \\ \beta = \frac{3\sqrt{|\Lambda|}}{(1+\alpha^2)^{3/2}}\mu\Xi \qquad &\Longleftrightarrow \qquad m := \Xi\mu = \frac{(1+\alpha^2)^{3/2}}{3\sqrt{|\Lambda|}}\beta\,. \end{aligned}$$

Letting β_c be the value of β corresponding to μ_c, one finds

$$\begin{aligned} \beta_c = &\frac{\sqrt{-9+36\gamma+\sqrt{3}\sqrt{(3+4\gamma)^3}}}{3\sqrt{2}} \\ \Longleftrightarrow \quad m_c^2 = &\frac{(1+\alpha^2)^3\left(-9+36\gamma+\sqrt{3}\sqrt{(3+4\gamma)^3}\right)}{162|\Lambda|}\,. \end{aligned} \tag{4.7.65}$$

When $q = 0$, the graph of β_c as a function of α can be found in Figure 4.7.10. In general, the graph of β_c as a function of a and q can be found in Figure 4.7.11.

Note that if $q = 0$, then γ can be used as a replacement for a; otherwise, γ is a substitute for q at fixed a.

When $e = 0$ we have $m_c = a + O(a^3)$ for small a, and $m_c \to \frac{8}{3\sqrt{|\Lambda|}}$ as $|a| \nearrow \sqrt{|3/\Lambda|}$.

According to [149], the physically relevant mass of the solution is μ and *not* m; because of the rescaling involved, we have $\mu_c \to \infty$ as $|a| \nearrow \sqrt{|3/\Lambda|}$.

We have $d^2\Delta_r/dr^2 > 0$, so that the set $\{\Delta_r \leq 0\}$ is an interval (r_-, r_+), with $0 < r_- < r_+$.

It follows from (4.7.44) that $g_{\varphi\varphi}/\sin^2(\theta)$ is strictly positive for $r > 0$, and the analysis of the time-machine set is identical to the case $\Lambda > 0$ as long as

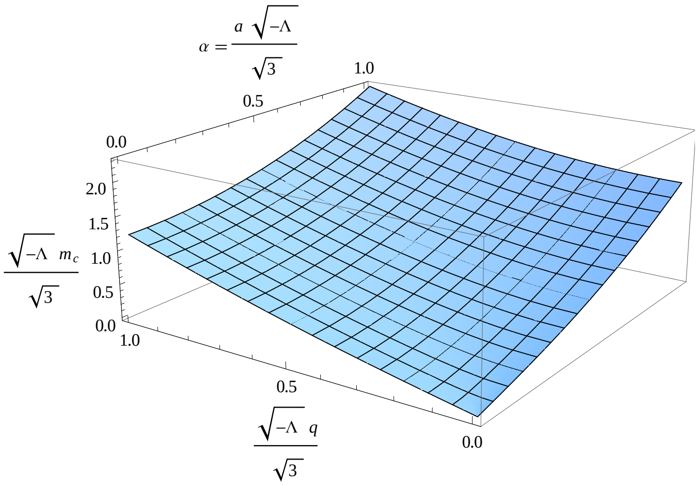

Figure 4.7.11: The critical mass parameter $m_c\sqrt{\frac{|\Lambda|}{3}}$ as a function of $\alpha = a\sqrt{\frac{|\Lambda|}{3}}$ and $q\sqrt{\frac{|\Lambda|}{3}}$.

$\Xi > 0$, which is assumed. We note that stable causality of each region on which Δ_r has constant sign follows from (4.7.37) or (4.7.38).

The projection metric is formally identical to that derived in Section 4.7.5, with projection diagrams as in Figure 4.7.12.

4.7.8 The Emparan-Reall metrics

We consider the Emparan-Reall black-ring metric as presented in [114]:

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dt - C\,R\,\frac{1+y}{F(y)}\,d\psi\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}\,F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right], \end{aligned} \tag{4.7.66}$$

where

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \tag{4.7.67}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{4.7.68}$$

The parameter λ is chosen to be

$$\lambda = \frac{2\nu}{1+\nu^2}\,, \tag{4.7.69}$$

with the parameter ν lying in $(0,1)$, so that

$$0 < \nu < \lambda < 1\,. \tag{4.7.70}$$

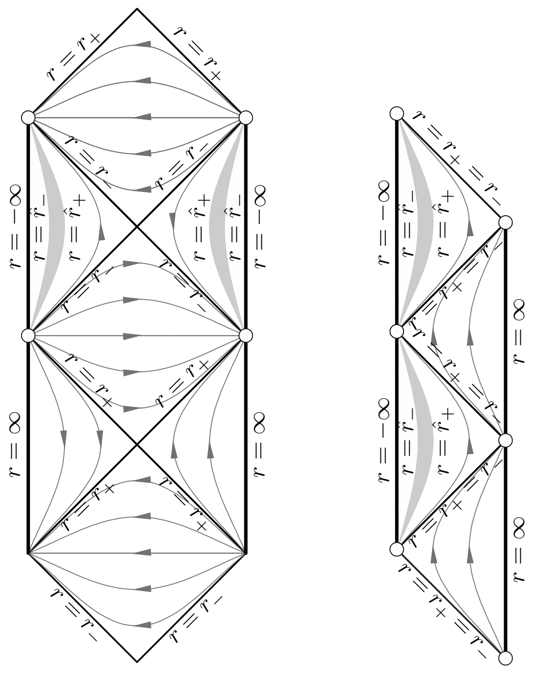

Figure 4.7.12: The projection diagrams for the Kerr-Newman - anti de Sitter metrics with two distinct zeros of Δ_r (left diagram) and one double zero (right diagram); see Remark 4.7.2.

The coordinates x, y lie in the ranges $-\infty \le y \le -1$, $-1 \le x \le 1$, assuming further that $(x,y) \neq (-1,-1)$. The event horizons are located at $y = y_h = -1/\nu$ and the ergosurface is at $y = y_e = -1/\lambda$. The ∂_ψ–axis is at $y = -1$ and the ∂_ϕ–axis is split into two parts $x = \pm 1$. Spatial infinity i^0 corresponds to $x = y = -1$. The metric becomes singular as $y \to -\infty$.

Although this is not immediately apparent from the current form of the metric, it is known [112] that ∂_ψ is spacelike or vanishing in the region of interest, with $g_{\psi\psi} > 0$ away from the rotation axis $y = -1$. Now, the metric (4.7.66) may be rewritten in the form

$$\begin{aligned} g &= \left(g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}}\right) dt^2 - \frac{R^2}{(x-y)^2}\frac{F(x)}{G(y)} dy^2 \\ &\quad \underbrace{+\, g_{\psi\psi}\left(d\psi + \frac{g_{t\psi}}{g_{\psi\psi}}dt\right)^2 + g_{xx}dx^2 + g_{\phi\phi}d\phi^2}_{\geq 0} \,. \end{aligned} \tag{4.7.71}$$

We have

$$g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}} = -\frac{G(y)F(y)F(x)}{F(x)^2 G(y) + C^2(1+y)^2(x-y)^2} \,. \tag{4.7.72}$$

It turns out that there is a non-obvious factorization of the denominator as

$$F(x)^2 G(y) + C^2(1+y)^2(x-y)^2 = -F(y) I(x,y) \,,$$

where I is a second-order polynomial in x and y with coefficients depending upon ν, sufficiently complicated so that it cannot be usefully displayed here. The polynomial I turns out to be non-negative, which can be seen using a trick similar to one in [92], as follows: One introduces new, non-negative, variables and parameters (X, Y, σ) via the equations

$$x = X - 1 \,, \quad y = -Y - 1 \,, \quad \nu = \frac{1}{1+\sigma} \,, \tag{4.7.73}$$

with $0 \le X \le 2$, $0 \le Y < +\infty$, $0 < \sigma < +\infty$. A Mathematica calculation shows that in this parameterization the function I is a rational function of the new variables, with a simple denominator which is explicitly non-negative, while the numerator is a complicated polynomial in X, Y, σ with, however, *all coefficients positive.*

Let $\Omega = (x-y)/\sqrt{F(x)}$, then the function

$$\kappa(x,y) := \Omega^2 \left(g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}}\right) = -\frac{G(y)F(y)}{\frac{F(x)^2}{(x-y)^2} G(y) + C^2(1+y)^2} \tag{4.7.74}$$

has extrema in x only for $x = y = -1$ and $x = -1/\lambda < -1$. This may be seen from its derivative with respect to x, which is explicitly non-positive in the ranges of variables of interest:

$$\frac{\partial \kappa}{\partial x} = -\frac{2G(y)^2 F(y)^2 F(x)(x-y)}{(F(x)^2 G(y) + C^2(1+y)^2(x-y)^2)^2} = -\frac{2G(y)^2 F(x)(x-y)}{I(x,y)^2} \,. \tag{4.7.75}$$

Therefore,

$$\frac{(1+y)^2G(y)}{I(-1,y)} = \kappa(-1,y) \geq \kappa(x,y) \geq \kappa(1,y) = \frac{(1-y)^2G(y)}{I(1,y)} .$$

Since both $I(-1,y)$ and $I(1,y)$ are positive, in the domain of outer communications $\{-1/\nu < y \leq -1\}$ where $G(y)$ is negative we obtain

$$\frac{-G(y)(1+y)^2}{I(-1,y)} \leq \left|\Omega^2\left(g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}}\right)\right| \leq \frac{-G(y)(1-y)^2}{I(1,y)} . \tag{4.7.76}$$

One finds

$$I(1,y) = \frac{1+\lambda}{1-\lambda}(-1+y^2)(1-y(\lambda-\nu)-\lambda\nu) ,$$

which leads to the projection metric

$$\gamma := \chi(y)\frac{G(y)}{(-1-y)}dt^2 - \frac{R^2}{G(y)}dy^2 , \tag{4.7.77}$$

where, using the variables (4.7.73) to make manifest the positivity of χ in the range of variables of interest,

$$\begin{aligned} \chi(y) &= \frac{(1-y)(1-\lambda)}{(1+\lambda)(1-y(\lambda-\nu)-\lambda\nu)} \\ &= \frac{(2+Y)\sigma(1+\sigma)(2+2\sigma+\sigma^2)}{(2+\sigma)^3(2+Y+\sigma)} > 0 . \end{aligned}$$

The calculation of (4.1.4) leads to the following conformal metric

$$\overset{(2)}{g} = R\sqrt{\frac{\chi}{|1+y|}}\left(-\hat{F}dt^2 + \hat{F}^{-1}dr^2\right) , \text{ where } \hat{F} = -\tfrac{1}{R}\sqrt{\tfrac{\chi}{|1+y|}}G . \tag{4.7.78}$$

Since the integral of $\hat{F}^{-1}$ diverges at the event horizon, and is finite at $y=-1$ (which corresponds *both* to an axis of rotation and the asymptotic region at infinity), the analysis in Sections 4.2.4 and 4.3 shows that the corresponding projection diagram is as in Figure 4.7.13.

It is instructive to compare this to the projection diagram for five-dimensional Minkowski spacetime

$$(t, \hat{r}\cos\phi, \hat{r}\sin\phi, \tilde{r}\cos\psi, \tilde{r}\sin\psi) \equiv (t, \hat{x}, \hat{y}, \tilde{x}, \tilde{y}) \in \mathbb{R}^5$$

parameterized by ring-type coordinates:

$$y = -\frac{\hat{r}^2}{(\hat{r}^2+\tilde{r}^2)^2} - 1 , \quad x = \frac{\tilde{r}^2}{(\hat{r}^2+\tilde{r}^2)^2} - 1 , \quad \hat{r} = \sqrt{\hat{x}^2+\hat{y}^2} , \quad \tilde{r} = \sqrt{\tilde{x}^2+\tilde{y}^2} .$$

For fixed $x \neq 0$, $y \neq 0$ we obtain a torus as φ and ψ vary over S^1. The image of the resulting map is the set $x \geq -1$, $y \leq -1$, $(x,y) \neq (-1,-1)$. Since

$$x - y = \frac{1}{\hat{r}^2+\tilde{r}^2} ,$$

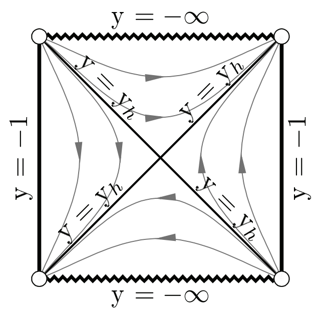

Figure 4.7.13: The projection diagram for the Emparan-Reall black rings. The arrows indicate the causal character of the orbits of the isometry group. The boundary $y = -1$ is covered, via the projection map, by the axis of rotation *and* by spatial infinity i^0. Curves approaching the conformal null infinities $\mathscr{I}^{\pm}$ asymptote to the missing corners in the diagram.

the spheres $\hat{r}^2+\tilde{r}^2 =: r^2 = \text{const}$ are mapped to subsets of the lines $x = y+1/r^2$, and the limit $r \to \infty$ corresponds to $0 \le x-y \to 0$ (hence $x \to -1$ and $y \to -1$). The inverse transformation reads

$$\hat{r} = \frac{\sqrt{-y-1}}{x-y}\,, \quad \tilde{r} = \frac{\sqrt{x+1}}{x-y}\,.$$

The Minkowski metric takes the form

$$\begin{aligned}
\eta &= -dt^2 + d\hat{x}^2 + d\hat{y}^2 + d\tilde{x}^2 + d\tilde{y}^2 \\
&= -dt^2 + d\hat{r}^2 + \hat{r}^2 d\varphi^2 + d\tilde{r}^2 + \tilde{r}^2 d\psi^2 \\
&= -dt^2 + \frac{dy^2}{4(-y-1)(x-y)^2} + \underbrace{\frac{dx^2}{4(x+1)(x-y)^2} + \hat{r}^2 d\varphi^2 + \tilde{r}^2 d\psi^2}_{\ge 0}\,.
\end{aligned}$$

Thus, for any η-causal vector X,

$$\eta(X,X) \;\ge\; -(X^t)^2 + \frac{(X^y)^2}{4(-y-1)(x-y)^2}\,.$$

There is a problem with the right-hand side since, at fixed y, x is allowed to go to infinity, and so there is no strictly positive lower bound on the coefficient of $(X^y)^2$. However, if we restrict attention to the set

$$r = \sqrt{\hat{r}^2 + \tilde{r}^2} \ge R$$

for some $R > 0$, we obtain

$$\eta(X,X) \;\ge\; -(X^t)^2 + \frac{R^4 (X^y)^2}{4(-y-1)}\,.$$

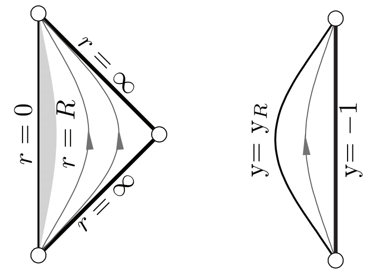

Figure 4.7.14: The projection diagram for the complement of a world-tube $\mathbb{R} \times B(R)$ in five-dimensional Minkowski spacetime using spherical coordinates (left figure, where the shaded region has to be removed), or using ring coordinates (right figure). In the right figure the right boundary $y = -1$ is covered, via the projection map, both by the axis of rotation and by spatial infinity, while null infinity projects to the missing points at the top and at the bottom of the diagram.

This leads to the conformal projection metric, for $-1 - \frac{1}{R^2} =: y_R \leq y \leq -1$,

$$\begin{aligned}
\gamma &:= -dt^2 + \frac{R^4 dy^2}{4|y+1|} \\
&= -dt^2 + \left(d\left(R^2\sqrt{|y+1|}\right)\right)^2 \\
&= \frac{R^2}{2\sqrt{|y+1|}}\left(-\frac{2\sqrt{|y+1|}}{R^2}dt^2 + \frac{R^2}{2\sqrt{|y+1|}}dy^2\right). \qquad (4.7.79)
\end{aligned}$$

Introducing a new coordinate $y' = -R^2\sqrt{-y-1}$ we have

$$\gamma = -dt^2 + dy'^2 \,,$$

where $-1 \leq y' \leq 0$. Therefore, the projection diagram corresponds to a subset of the standard diagram for a two-dimensional Minkowski spacetime, see Figure 4.7.14.

4.7.9 The Pomeransky-Senkov metrics

We consider the *Pomeransky-Senkov metrics* [236],

$$\begin{aligned}
g &= \frac{2H(x,y)k^2}{(1-\nu)^2(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) - 2\frac{J(x,y)}{H(y,x)}d\varphi d\psi \\
&\quad -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 + \frac{F(y,x)}{H(y,x)}d\varphi^2 \,, \qquad (4.7.80)
\end{aligned}$$

where Ω is a 1-form given by

$$\Omega = M(x,y)d\psi + P(x,y)d\varphi \,.$$

The definitions of the metric functions may be found in [236].[5] The metric depends on three constants: k, ν, λ, where k is assumed to be in $\mathbb{R}^*$, while the parameters λ and ν are restricted to the set[6]

$$\{(\nu, \lambda) : \ \nu \in (0,1)\,,\ 2\sqrt{\nu} \le \lambda < 1+\nu\}\,. \tag{4.7.81}$$

The coordinates x, y, φ, ψ, and t vary within the ranges $-1 \le x \le 1$, $-\infty < y < -1$, $0 \le \varphi \le 2\pi$, $0 \le \psi \le 2\pi$ and $-\infty < t < \infty$.

A Cauchy horizon is located at

$$y_c := -\frac{\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu}\,,$$

and the event horizon corresponds to

$$y_h := -\frac{\lambda - \sqrt{\lambda^2 - 4\nu}}{2\nu}\,.$$

Using an appropriate Gauss diagonalization, the metric may be rewritten in the form

$$\begin{aligned} g = {} & \frac{\overbrace{g_{t\psi}^2 g_{\varphi\varphi} - 2g_{t\varphi}g_{t\psi}g_{\psi\varphi} + g_{t\varphi}^2 g_{\psi\psi} + g_{tt}(g_{\psi\varphi}^2 - g_{\varphi\varphi}g_{\psi\psi})}^{(*)}}{g_{\psi\varphi}^2 - g_{\varphi\varphi}g_{\psi\psi}}\, dt^2 + g_{yy}dy^2 \\ & \underbrace{+ g_{xx}dx^2 + \left(g_{\varphi\varphi} - \frac{g_{\psi\varphi}^2}{g_{\psi\psi}}\right)\left(d\varphi + \frac{g_{t\varphi} - \frac{g_{t\psi}g_{\psi\varphi}}{g_{\psi\psi}}}{g_{\varphi\varphi} - \frac{g_{\psi\varphi}^2}{g_{\psi\psi}}}dt\right)^2 + \frac{(g_{t\psi}dt + g_{\psi\varphi}d\varphi + g_{\psi\psi}d\psi)^2}{g_{\psi\psi}}}_{(**)}\,. \end{aligned} \tag{4.7.82}$$

The positive-definiteness of $(**)$ for $y > y_c$ follows from [69, 92]. Note that $g_{\psi\psi} < 0$ would give a timelike Killing vector ∂_ψ, and that $g_{\varphi\varphi}g_{\psi\psi} - g_{\psi\varphi}^2 < 0$ would lead to some combination of the periodic Killing vectors ∂_φ and ∂_ψ being timelike, so the term $(**)$ in (4.7.82) is non-negative on any region where there are no obvious causality violations.

The coefficient $(*)$ in front of dt^2 is negative for $y > y_h$ and positive for $y < y_h$, vanishing at $y = y_h$. This may be seen in the reparameterized form of the Pomeransky-Senkov solution that was introduced in [92]: Indeed, let a, b be the new coordinates as in [92] replacing x and y, respectively, and let us reparameterize ν, λ by c, d again as in [92], where all the variables a, b, c, d are non-negative above the Cauchy horizon, $y > y_c$:

$$\begin{aligned} x &= -1 + \frac{2}{1+a}\,, \\ y &= -1 - \frac{d(4+c+2d)}{(1+b)(2+c)}\,, \end{aligned}$$

[5] We use (ψ, φ) where Pomeransky & Senkov use (φ, ψ).

[6] $\nu = 0$ corresponds to Emparan-Reall metric which has been already analysed in Section 4.7.8.

$$\begin{aligned}\nu &= \frac{1}{(1+d)^2}\,,\\ \lambda &= 2\frac{2d^2+2(2+c)d+(2+c)^2}{(2+c)(1+d)(2+c+2d)}\,.\end{aligned} \tag{4.7.83}$$

Set

$$\kappa := (*)\,\Omega^2\,, \tag{4.7.84}$$

$$\Omega^2 := \frac{(x-y)^2(1-\nu)^2}{2k^2H(x,y)}\,. \tag{4.7.85}$$

Using MATHEMATICA one finds that κ takes the form

$$\kappa = -\Omega^2(y-y_h)Q\,,$$

where $Q = Q(a,b,c,d)$ is a huge rational function in (a,b,c,d) with *all coefficients positive.* To obtain the corresponding projection metric γ one would have, e.g., to find sharp lower and upper bounds for Q, at fixed y, which would lead to

$$\gamma := -(y-y_h)\sup_{\text{y fixed}}|Q|\;dt^2 - \frac{1}{G(y)}dy^2\,.$$

This requires analyzing a complicated rational function, which has not been done in the literature so far.

We expect the corresponding projection diagram to look like that for Kerr - anti de Sitter spacetime of Figure 4.7.12, with $r=\infty$ there replaced by $y=-1$, $r=-\infty$ replaced by $y=1$ with an appropriate analytic continuation of the metric to positive y's (compare [69]), r_+ replaced by y_h and r_- replaced by y_c. The shaded regions in the negative region there might be non-connected for some values of parameters, and always extend to the boundary at infinity in the relevant diamond [69].

Recall that a substantial part of the work in [69] was to show that the function $H(x,y)$ had no zeros for $y>y_c$. We note that the reparameterization

$$y \to -1 - \frac{cd}{(1+b)(2+c+2d)}$$

of [92] (with the remaining formulae (4.7.83) remaining the same), gives

$$H(x,y) = \frac{P(a,b,c,d)}{(1+a)^2(1+b)^2(2+c)^2(1+d)^6(2+c+2d)^4}\,,$$

where P is a huge polynomial with *all coefficients positive* for $y>y_h$. This establishes immediately positivity of $H(x,y)$ in the domain of outer communications. However, positivity of $H(x,y)$ in the whole range $y>y_c$ has only been established so far using the rather more involved analysis in [69].

4.8 Black holes vs. spatially compact U(1)×U(1) symmetric models with compact Cauchy horizons

It turns out that one use the Kerr-Newman - (a)dS family of metrics to construct explicit examples of maximal, four-dimensional, $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetric, electrovacuum or vacuum models, with or without cosmological constant, containing a spatially compact partial Cauchy surface. Similarly, five-dimensional, $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$ symmetric, spatially compact vacuum models with spatially compact partial Cauchy surfaces can be constructed using the Emparan-Reall or Pomeransky-Senkov metrics. We will show how the projection diagrams constructed so far can be used to understand maximal (non-globally hyperbolic) extensions of the maximal globally hyperbolic regions in such models, and for the Taub-NUT metrics.

4.8.1 Kerr-Newman-(a)dS-type and Pomeransky-Senkov-type models

The diamonds and triangles which have been used to construct our diagrams so far will be referred to as *blocs*. Here the notion of a triangle is understood *up to diffeomorphism*, thus planar sets with three corners, connected by smooth curves intersecting only at the corners which are *not* necessarily straight lines, are also considered to be triangles.

In the interior of each bloc one can periodically identify points lying along the orbits of the action of the $\mathbb{R}$ factor of the isometry group. Here we are only interested in the connected component of the identity of the group, which is $\mathbb{R}\times\mathrm{U}(1)$ in the four-dimensional case, and $\mathbb{R}\times\mathrm{U}(1)\times\mathrm{U}(1)$ in the five-dimensional case.

Note that isometries of spacetime extend smoothly across all bloc boundaries. For example, in the coordinates $(v, r, \theta, \tilde{\varphi})$ discussed in the paragraph around (4.7.20), p. 173, translations in t become translations in v; similarly for the $(u, r, \theta, \tilde{\varphi})$ coordinates. Using the $(U, V, \theta, \tilde{\varphi})$ local coordinates near the intersection of two Killing horizons, translations in t become boosts in the (U, V) plane.

Consider one of the blocs, out of any of the diagrams constructed above, in which the orbits of the isometry group are spacelike. (Note that no such diamond or triangle has a shaded area which needs to be excised, as the shadings occur only within those building blocs where the isometry orbits are timelike.) It can be seen that the periodic identifications result then in a spatially compact maximal globally hyperbolic spacetime with $S^1 \times S^2$ spatial topology, respectively with $S^1 \times S^1 \times S^2$ topology.

Now, each diamond in our diagrams has four null boundaries which naturally split into pairs, as follows: In each bloc in which the isometry orbits are spacelike, we will say that two boundaries are *orbit-adjacent* if both boundaries lie to the future of the bloc, or both to the past. In a bloc where the isometry orbits are timelike, boundaries will be said *orbit-adjacent* if they are both to the left or both to the right.

One out of each pair of orbit-adjacent null boundaries of a bloc with spacelike isometry-orbits corresponds, in the periodically identified spacetime, to a compact Cauchy horizon across which the spacetime can be continued to a periodically identified adjacent bloc. Which of the two adjacent boundaries will become a Cauchy horizon is a matter of choice; once such a choice has been made, the other boundary *cannot* be attached anymore: those geodesics which, in the

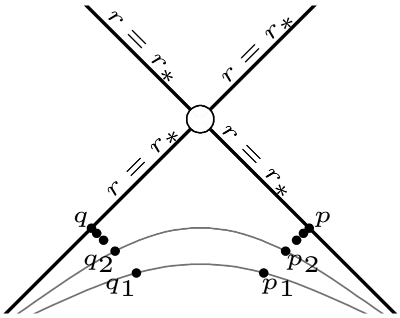

Figure 4.8.1: The sequences q_i and p_i. Rotating the figure by integer multiples of 90 degrees shows that the problem of non-unique limits arises on any pair of orbit-adjacent boundaries.

unidentified spacetime, would have been crossing the second boundary become, in the periodically identified spacetime, incomplete inextendible geodesics. This behaviour is well known from Taub-NUT spacetimes [78, 207, 264], and is easily seen as follows:

Consider a sequence of points $p_i := (t_i, r_i)$ such that p_i converges to a point p on a horizon in a projection diagram in which no periodic identifications have been made. Let $T > 0$ be the period with which the points are identified along the isometry orbits, thus for every $n \in \mathbb{Z}$ points (t, r) and $(t + nT, r)$ represent the same point of the quotient manifold. It should be clear from the form of the Eddington-Finkelstein type coordinates u and v used to perform the two distinct extensions (see the paragraph around (4.7.20), p. 173) that there exists a sequence $n_i \in \mathbb{Z}$ such that, passing to a subsequence if necessary, the sequence $q_i = (t_i + n_i T, r_i)$ converges to some point q in the companion orbit-adjacent boundary, see Figure 4.8.1.

Denote by $[p]$ the class of p under the equivalence relation $(t, r) \sim (t + nT, r)$, where $n \in \mathbb{Z}$ and T is the period. Suppose that one could construct simultaneously an extension of the quotient manifold across both orbit-adjacent boundaries. Then the sequence of points $[q_i] = [p_i]$ would have two distinct points $[p]$ and $[q]$ as limit points, which is not possible. This establishes our claim.

Returning to our main line of thought, note that a periodically identified building bloc in which the isometry orbits are timelike will have obvious causality violations throughout, as a linear combination of the periodic Killing vectors becomes timelike there.

The branching construction, where one out of the pair of orbit-adjacent boundaries is chosen to perform the extension, can be continued at each bloc in which the isometry orbits are spacelike. This shows that maximal extensions are obtained from any connected union of blocs such that in each bloc an extension is carried out across precisely one out of each pair of orbit-adjacent boundaries. Some such subsets of the plane might only comprise a finite number of blocs, as seen trivially in Figure 4.7.9. Clearly an infinite number of distinct finite, semi-infinite, or infinite sequences of blocs can be constructed in the diagram of

Figure 4.7.6. Two sequences of blocs which are not related by one of the discrete isometries of the diagram will lead to non-isometric maximal extensions of the maximal globally hyperbolic initial region.

4.8.2 Taub-NUT metrics

We have seen at the end of Section 4.7.2 how to construct a projection diagram for Gowdy cosmological models. Those models all contain $\mathrm{U}(1) \times \mathrm{U}(1)$ as part of their isometry group. The corresponding projection diagrams constructed in Section 4.7.2 were obtained by projecting-out the isometry orbits. This is rather different from the remaining projection diagrams constructed in this work, where only one of the coordinates along the Killing orbits was projected out.

It is instructive to carry out explicitly both procedures for the Taub-NUT metrics, which belong to the Gowdy class. Using Euler angles (ζ, θ, φ) to parameterize S^3, the Taub-NUT metrics [218, 264] take the form

$$g = -U^{-1}dt^2 + (2\ell)^2 U(d\zeta + \cos(\theta)\, d\varphi)^2 + (t^2+\ell^2)(d\theta^2 + \sin^2(\theta)\, d\varphi^2)\,. \quad (4.8.1)$$

Here

$$U(t) = -1 + \frac{2(mt+\ell^2)}{t^2+\ell^2} = \frac{(t_+ - t)(t - t_-)}{t^2+\ell^2}\,,$$

with

$$t_\pm := m \pm \sqrt{m^2+\ell^2}\,.$$

Further, ℓ and m are real numbers with $\ell > 0$. The region $\{t \in (t_-, t_+)\}$ will be referred to as the *Taub spacetime*.

The metric induced on the sections $\theta = \text{const}$, $\varphi = \text{const}'$, of the Taub spacetime reads

$$\gamma_0 := -U^{-1}dt^2 + (2\ell)^2 U d\zeta^2\,. \quad (4.8.2)$$

As discussed by Hawking and Ellis [147], this is a metric to which the methods of Section 4.1 apply *provided that the 4π-periodic identifications in ζ are relaxed.* Since U has two simple zeros, and no singularities, the conformal diagram for the corresponding maximally extended two-dimensional spacetime equipped with the metric γ_0 coincides with the left diagram in Figure 4.8.2, compare [147, Figure 33]. The discussion of the last paragraph of the previous section applies and, together with the left diagram in Figure 4.8.2, provides a family of simply connected maximal extensions of the sections $\theta = \text{const}$, $\varphi = \text{const}'$, of the Taub spacetime.

However, it is not clear how to relate the above to extensions of the four-dimensional spacetime. Note that projecting out the ζ and φ variables in the region where $U > 0$, using the projection map $\pi_1(t,\zeta,\theta,\varphi) := (t,\theta)$, one is left with the two-dimensional metric

$$\gamma_1 := -U^{-1}dt^2 + (t^2+\ell^2)\, d\theta^2\,, \quad (4.8.3)$$

which leads to the flat metric on the Gowdy square as the projection metric. (The coordinate t here is not the same as the Gowdy t coordinate, but the

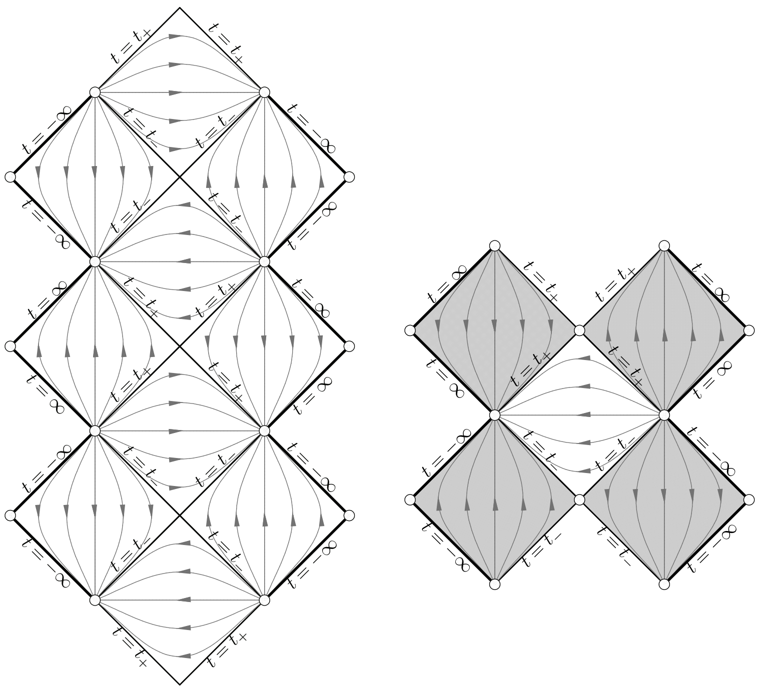

Figure 4.8.2: The left diagram is the conformal diagram for an extension of the universal covering space of the sections $\theta = \text{const}$, $\varphi = \text{const}'$, of the Taub spacetime. The right diagram represents simultaneously the four possible diagrams for the maximal extensions, within the Taub-NUT class, with compact Cauchy horizons, of the Taub spacetime. After invoking the left-right symmetry of the diagram, which lifts to an isometry of the extended spacetime, the four diagrams lead to two non-isometric spacetimes.

projection diagram remains a square.) And one is left wondering how this fits with the previous picture.

Now, one can attempt instead to project out the θ and φ variables, with the projection map

$$\pi_2(t,\zeta,\theta,\varphi) := (t,\zeta)\,. \tag{4.8.4}$$

For this we note the trivial identity

$$g_{\zeta\zeta}d\zeta^2 + 2g_{\varphi\zeta}d\varphi\, d\zeta + g_{\varphi\varphi}d\varphi^2 = (g_{\zeta\zeta} - \frac{g_{\varphi\zeta}^2}{g_{\varphi\varphi}})d\zeta^2 + \underbrace{g_{\varphi\varphi}(d\varphi + \frac{g_{\varphi\zeta}}{g_{\varphi\varphi}}d\zeta)^2}_{(*)}\,. \tag{4.8.5}$$

Since the left-hand side is positive-definite on Taub space, where $U > 0$, both $g_{\zeta\zeta} - \frac{g_{\varphi\zeta}^2}{g_{\varphi\varphi}}$ and $g_{\varphi\varphi}$ are non-negative there. Indeed,

$$g_{\varphi\varphi} = \left(\ell^2 + t^2\right)\sin^2(\theta) + 4\ell^2 U\cos^2(\theta)\,, \tag{4.8.6}$$

$$\begin{aligned} g_{\zeta\zeta} - \frac{g_{\varphi\zeta}^2}{g_{\varphi\varphi}} &= (2\ell)^2\left(1 - \frac{(2\ell)^2 U\cos^2(\theta)}{g_{\varphi\varphi}}\right)U \\ &= \underbrace{\frac{4\ell^2\left(\ell^2+t^2\right)\sin^2(\theta)}{(\ell^2+t^2)\sin^2(\theta) + 4\ell^2 U\cos^2(\theta)}}_{(**)}U\,. \end{aligned} \tag{4.8.7}$$

However, perhaps not unsurprisingly given the character of the coordinates involved, the function $(**)$ in (4.8.7) does not have a positive lower bound independent of $\theta \in [0, 2\pi]$, which is unfortunate for our purposes. To sidestep this drawback we choose a number $0 < \epsilon < 1$ and restrict ourselves to the range $\theta \in [\theta_\epsilon, \pi - \theta_\epsilon]$, where $\theta_\epsilon \in [0, \pi/2]$ is defined by

$$\sin^2(\theta_\epsilon) = \epsilon\,.$$

Now, $g_{\varphi\varphi}$ is strictly positive for large t, independently of θ. Next, $g_{\varphi\varphi}$ equals $4\ell^2 U$ at the axes of rotation $\sin(\theta) = 0$, and equals $\ell^2 + t^2$ at $\theta = \pi/2$. Hence, keeping in mind that U is monotonic away from (t_-, t_+), for ϵ small enough there will exist values

$$\hat t_\pm(\epsilon)\,,\ \text{with } \hat t_-(\epsilon) < t_- < 0 < t_+ < \hat t_+(\epsilon)$$

such that $g_{\varphi\varphi}$ will be negative somewhere in the region $(\hat t_-(\epsilon), t_-) \cup (t_+, \hat t_+(\epsilon))$, and will be positive outside of this region. We choose those numbers to be optimal with respect to those properties.

On the other hand, for ϵ close enough to 1 the metric coefficient $g_{\varphi\varphi}$ will be strictly positive for all $\theta \in [\theta_\epsilon, \pi - \theta_\epsilon]$ and $t < t_-$. In this case we set $\hat t_-(\epsilon) = t_-$, so that the interval $(\hat t_-(\epsilon), t_-)$ is empty. Similarly, there will exist a range of ϵ for which $\hat t_+(\epsilon) = t_+$, and $(t_+, \hat t_+(\epsilon)) = \emptyset$. The relevant ranges of ϵ will coincide only if $m = 0$.

We note

$$\partial_\theta\left(g_{\zeta\zeta} - \frac{g_{\varphi\zeta}^2}{g_{\varphi\varphi}}\right) = \frac{16\ell^4 U^2\left(\ell^2+t^2\right)\sin(2\theta)}{\left((\ell^2+t^2)\sin^2(\theta) + 4\ell^2 U\cos^2(\theta)\right)^2}\,,$$

which shows that, for

$$t \notin (\hat{t}_-(\epsilon), t_-) \cup (t_+, \hat{t}_+(\epsilon)) \text{ and } \theta \in (\theta_\epsilon, \pi - \theta_\epsilon), \tag{4.8.8}$$

the multiplicative coefficient $(**)$ of U in (4.8.7) will satisfy

$$(**) \geq \frac{4\ell^2 \left(\ell^2 + t^2\right) \sin^2(\theta_\epsilon)}{(\ell^2 + t^2) \sin^2(\theta_\epsilon) + 4\ell^2 U \cos^2(\theta_\epsilon)} =: f_\epsilon(t)\,. \tag{4.8.9}$$

We are ready now to construct the projection metric in the region (4.8.8). Removing from the metric tensor (4.8.1) the terms $(*)$ appearing in (4.8.5), as well as the $d\theta^2$ terms, and using (4.8.9) one finds, for g-causal vectors X,

$$g(X, X) \geq \gamma_2((\pi_2)_* X, (\pi_2)_* X)\,,$$

with π_2 as in (4.8.4), and where

$$\gamma_2 := -U^{-1} dt^2 + f_\epsilon\, U d\zeta^2\,. \tag{4.8.10}$$

Since U has exactly two simple zeros and is finite everywhere, and for ϵ such that $g_{\varphi\varphi}$ is positive on the region $\theta \in [\theta_\epsilon, \pi - \theta_\epsilon]$, the projection diagram for that region, *in a spacetime in which no periodic identifications in ζ are made*, is given by the left diagram of Figure 4.8.2. The reader should have no difficulties finding the corresponding diagrams for the remaining values of ϵ.

However, we are in fact interested in those spacetimes where ζ is 4π periodic. This has two consequences: a) there are closed timelike Killing orbits in all the regions where U is negative, and b) no simultaneous extensions are possible across two orbit-adjacent boundaries. It then follows (see the right diagram of Figure 4.8.2) that there are, within the Taub-NUT class, only two non-isometric, maximal, vacuum extensions across compact Cauchy horizons of the Taub spacetime. (Compare [56, Proposition 4.5 and Theorem 1.2] for the local uniqueness of extensions, and [66] for a discussion of extensions with non-compact Killing horizons.)

Chapter 5

Alternative approaches

In most of our discussion so far we have been considering stationary spacetimes. In those it is most convenient to define the black hole region using the flow of the Killing vector field, as presented in Section 1.3.7, p. 54. In non-stationary spacetimes this does not work, and a different approach is needed. The standard way to do this invokes conformal completions, which we critically review in Section 5.1 below. We then pass to a discussion of alternative possibilities.

5.1 The standard approach and its shortcomings

The standard way of defining black holes is by using conformal completions: A pair $(\widetilde{\mathscr{M}}, \widetilde{g})$ is called a *conformal completion* of $(\mathscr{M}, g)$ if $\widetilde{\mathscr{M}}$ is a manifold with boundary such that:

1. $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$,
2. there exists a function Ω, with the property that the metric $\widetilde{g}$, defined to be $\Omega^2 g$ on $\mathscr{M}$, extends by continuity to the boundary of $\widetilde{\mathscr{M}}$, with the extended metric still being non-degenerate throughout,
3. Ω is positive on $\mathscr{M}$, differentiable on $\widetilde{\mathscr{M}}$, vanishes on $\mathscr{I}$, with $d\Omega$ *nowhere vanishing on* $\mathscr{I}$.

We emphasize that no assumptions about the causal nature of Scri are made so far. The boundary of $\widetilde{\mathscr{M}}$ will be called Scri, denoted $\mathscr{I}$.

In the standard treatments of the problem at hand [147, 271] smoothness of both the conformal completion and the metric $\widetilde{g}$ is imposed, though this can be weakened for many purposes.

Let $(\widetilde{\mathscr{M}}, \widetilde{g})$ be a conformal completion at infinity of $(\mathscr{M}, g)$. One sets

$$\mathscr{I}^+ = \{p \in \mathscr{I} \mid I^-(p; \widetilde{\mathscr{M}}) \cap \mathscr{M} \neq \emptyset\} .$$

Assuming various global regularity conditions on the conformal completion $\widetilde{\mathscr{M}}$, the black hole region $\mathscr{B}$ is then defined as (cf., e.g., [147, 271])

$$\mathscr{B} := \mathscr{M} \setminus J^-(\mathscr{I}^+) . \tag{5.1.1}$$

Let us point out some drawbacks of this approach:

- **Non-equivalent Scri's:** Conformal completions at null infinity do not have to be unique, an example can be constructed as follows:

  The Taub–NUT metrics can be locally written in the form [207]

$$-U^{-1}dt^2 + (2L)^2 U\sigma_1^2 + (t^2 + L^2)(\sigma_2^2 + \sigma_3^2)\,, \qquad (5.1.2)$$

$$U(t) = -1 + \tfrac{2(mt+L^2)}{t^2+L^2}\,. \qquad (5.1.3)$$

  where σ_1, σ_2 and σ_3 are left invariant one–forms on $SU(2) \approx S^3$. The constants L and m are real numbers with $L > 0$. Parameterizing S^3 with Euler angles (μ, θ, φ) one is led to the following form of the metric

$$g = -U^{-1}dt^2 + (2L)^2 U(d\mu + \cos\theta d\varphi)^2 + (t^2 + L^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,.$$

  To construct the conformal completions, one first passes to a coordinate system which is used to perform extensions across the Cauchy horizons $t_\pm := M \pm \sqrt{M^2 + L^2}$:

$$(t, \mu, \theta, \varphi) \to (t, \mu \pm \int_{t_0}^{t} [2LU(s)]^{-1} ds, \theta, \varphi)\,. \qquad (5.1.4)$$

  Denoting by $g_\pm$ the metric g in the new coordinates, one finds

$$\begin{aligned} g_\pm &= \pm 4L(d\mu + \cos\theta d\varphi)dt \\ &\quad +(2L)^2 U(d\mu + \cos\theta d\varphi)^2 + (t^2 + L^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \qquad (5.1.5)$$

  Each of the metrics $g_\pm$ can be smoothly conformally extended to the boundary at infinity "$t = \infty$" by introducing

$$x = 1/t\,,$$

  so that (5.1.5) becomes

$$\begin{aligned} g_\pm &= x^{-2}\Big(\mp 4L(d\mu + \cos\theta d\varphi)dx \\ &\quad +(2L)^2 x^2 U(d\mu + \cos\theta d\varphi)^2 + (1 + L^2x^2)(d\theta^2 + \sin^2\theta d\varphi^2)\Big)\,. \end{aligned} \qquad (5.1.6)$$

  In each case this leads to a Scri diffeomorphic to S^3. There is a simple isometry between g_+ and g_- given by

$$(x, \mu, \theta, \varphi) \to (x, -\mu, \theta, -\varphi)$$

  (this does correspond to a smooth map of the region $t \in (t_+, \infty)$ into itself, cf. [78]), so that the two Scri's so obtained are isometric. However, in addition to the two ways of attaching Scri to the region $t \in (t_+, \infty)$ there are the two corresponding ways of extending this region across the Cauchy horizon $t = t_+$, leading to four possible manifolds with boundary. It can then be seen, using e.g. the arguments of [78], that the four possible manifolds split into two pairs, each of the manifolds from one pair *not* being isometric to one from the other. Taking into account the corresponding completion at "$t = -\infty$", and the two extensions across the Cauchy horizon $t = t_-$, one is led to four inequivalent conformal completions of each of the two inequivalent [78] time-oriented, maximally extended, standard Taub-NUT spacetimes.

This example naturally raises the question, *how many different conformal completions can a spacetime have?* Under mild completeness conditions in the spirit of [133], uniqueness of $\mathscr{I}^+$ as a point set in *past-distinguishing* spacetimes should follow from the TIP and TIF construction of [134]; note that this last condition is not satisfied by the Taub-NUT example.

The TIP/TIF analysis does, however, not carry information about differentiability. It turns out that, building upon an approach proposed by Geroch [132], one can prove existence and uniqueness of *maximal* strongly causal conformal completions in the smooth category, *provided there exists a non-trivial one* [66, Theorem 5.3] (compare [132, Theorem 2, p. 14]). But note that the existence of a non-empty strongly causal completion seems to be difficult to control in general situations.[1] In particular it could happen that many spacetimes of interest admitting conformal completions do not admit any strongly causal ones.

We note that uniqueness of a class of Riemannian conformal completions at infinity has been established in [77, Section 6], based on the analysis in [85]; this result can probably be used to obtain uniqueness of differentiable structure of Lorentzian conformal completions for Scri's admitting cross-sections, but we have not attempted to explore this idea any further.

Further partial results on the problem at hand can be found in [251].

- **Poorly differentiable Scri's:** In all standard treatments [132, 147, 271] it is assumed that both the conformal completion $\widetilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$ and the extended metric $\widetilde{g}$ are smooth, or have a high degree of differentiability [234]. This is a restriction which excludes most spacetimes which are asymptotically Minkowskian in lightlike directions, see [**?**, 88, **?**, 175] and references therein. Poor differentiability properties of $\mathscr{I}$ change the peeling properties of the gravitational field [81], but most – if not all – essential properties of black holes should be unaffected by conformal completions with, e.g., polyhomogeneous differentiability properties as considered in [10, 81]. It should, however, be borne in mind that the hypothesis of smoothness has been done in the standard treatments, so that in a complete theory the validity of various claims should be reexamined.

  A breakthrough result concerning the asymptotics at Scri is due to Hintz and Vasy [151], who prove that initial data which are polyhomogeneous at spatial infinity lead to spacetimes with a polyhomogeneous Scri. Further results concerning existence of spacetimes with a poorly differentiable Scri can be found in [95, 185].

[1] Globally hyperbolic conformal completions (in the sense of manifolds with boundary) are necessarily strongly causal. But it should be borne in mind that good causal properties of a spacetime might fail to survive the process of adding a conformal boundary.

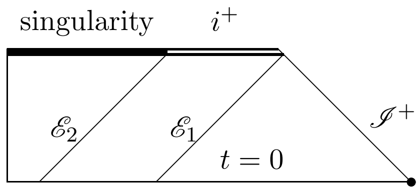

Figure 5.1.1: An asymptotically flat spacetime with an unusual i^+.

- **The structure of i^+:** The current theory of black holes is entirely based on intuitions originating in the Kerr and Schwarzschild geometries. In those spacetimes we have a family of preferred "stationary" observers which follow the orbits of the Killing vector field ∂_t in the asymptotic region, and their past coincides with that of $\mathscr{I}^+$. It is customary to denote by i^+ the set consisting of the points $t = \infty$, where t is the Killing time parameter for those observers. Now, the usual conformal diagrams for those spacetimes [147, 208] leave the highly misleading impression that i^+ is a regular point in the conformally rescaled manifold, which, to the best of our knowledge, is not the case. In dynamical cases the situation is likely to become worse. For example, one can imagine black hole spacetimes with a conformal diagram which, to the future of a Cauchy hypersurface $t = 0$, looks as in Figure 5.1.1. In that diagram the set i^+ should be thought of as the addition to the spacetime manifold $\mathscr{M}$ of a set of points "$\{t = \infty, q \in \mathscr{O}\}$", where $t \in [0, \infty)$ is the proper time for a family of observers $\mathscr{O}$. The part of the boundary of $\widetilde{\mathscr{M}}$ corresponding to i^+ is a singularity of the conformally rescaled metric, but we assume that it does *not* correspond to singular behaviour in the physical spacetime. In this spacetime there is the usual event horizon $\mathscr{E}_1$ corresponding to the boundary of the past of $\mathscr{I}^+$, which is completely irrelevant for the family of observers $\mathscr{O}$, and an event horizon $\mathscr{E}_2$ which is the boundary of the *true* black hole region for the family $\mathscr{O}$, i.e., the region that is not accessible to observations for the family $\mathscr{O}$. Clearly the usual black hole definition carries no physically interesting information in such a setting.

- **Causal regularity of Scri:** As already pointed out, in order to be able to prove interesting results the definition (5.1.1) should be complemented by causal conditions on $\widetilde{\mathscr{M}}$. The various approaches to this question are aesthetically highly unsatisfactory: it appears reasonable to impose causal regularity conditions on a spacetime, but why should some unphysical completion have any such properties? Clearly, the physical properties of a black hole should not depend upon the causal regularity – or lack thereof – of some artificial boundary which is being attached to the spacetime. While it seems reasonable and justified to restrict attention to spacetimes which possess good causal properties, it is not clear why the addition of artificial boundaries should preserve those properties, or even be consistent with them. Physically motivated restrictions are relevant when dealing with physical objects, they are not when non-physical constructs are considered.

- **Inadequacy for numerical purposes:** Most[2] numerical studies of black holes are performed on numerical grids which cover finite spacetime regions. Clearly, it would be convenient to have a set-up which is more compatible in spirit with such calculations than the Scri one.

We present in Sections 5.2.1 and 5.2.2 below two approaches in which the above listed problems are avoided.

5.2 Black holes without Scri

There has been considerable progress in the numerical analysis of black hole solutions of Einstein's equations; here one of the objectives is to write a stable code which would solve the full four dimensional Einstein equations, with initial data containing a non-connected black hole region that eventually merges into a connected one. One wishes to be able to consider initial data which do not possess any symmetries, and which have various parameters – such as the masses of the individual black holes, their angular momenta, as well as distances between them – which can be varied in significant ranges. Finally one wishes the code to run to a stage where the solution settles to a state close to equilibrium. The challenge then is to calculate the gravitational wave forms for each set of parameters, which could then be used in the gravitational wave observatories to determine the parameters of the collapsing black holes out of the observations made. This program has been being undertaken for years by several groups of researchers, with steady progress being made [5, 6, 12, 16, 29, 33, 140, 143, 169, 183, 237].[3]

There is a fundamental difficulty above, of deciding whether or not one is dealing indeed with the desired black hole initial data: the definition (5.1.1) of a black hole requires a conformal boundary $\mathscr{I}$ satisfying some properties. Clearly there is no way of ensuring those requirements in a calculation performed on a finite spacetime grid.[4]

In practice what one does is to set up initial data on a finite grid so that the region near the boundary is close to flat (in the conformal approach the whole asymptotically flat region is covered by the numerical grid, and does not

[2]The numerical simulations in [16, 29, 140] cover regions extending all the way to infinity, within frameworks which seem to be closely related to the "naive" framework of Section 5.2.1 below.

[3]Some spectacular visualizations of the calculations performed can be found at the URL `http://jean-luc.aei.mpg.de/NCSA1999/GrazingBlackHoles`

[4]The conformal approach developed by Friedrich (cf., e.g., [121, 122] and references therein) provides an ideal numerical framework for studying gravitational radiation in situations where the extended spacetime is smoothly conformally compactifiable across i^+, since then one can hope that the code will be able to "calculate Scri" globally to the future of the initial hyperboloidal hypersurface. It is not clear whether a conformal approach could provide more information than the non-conformal ones when i^+ is itself a singularity of the conformally rescaled equations, as is the case for black holes.

need to be near the boundary of the numerical grid; this distinction does not affect the discussion here). Then one evolves the initial data as long as the code allows. The gravitational waves emitted by the system are then extracted out of the metric near the boundary of the grid. Now, our understanding of energy emitted by gravitational radiation is essentially based on an analysis of the metric in an asymptotic region where g is nearly flat. In order to recover useful information out of the numerical data it is thus necessary for the metric near the boundary of the grid to remain close to a flat one. If we want to be sure that the information extracted contains all the essential dynamical information about the system, the metric near the boundary of the grid should quiet down to an almost stationary state as time evolves. Now, it is straightforward to set-up a mathematical framework to describe such situations without having to invoke conformal completions, this is done in the next section.

5.2.1 Naive black holes

Consider a *globally hyperbolic* spacetime $\mathscr{M}$ which contains a region covered by coordinates (t, x^i) with ranges

$$r := \sqrt{\sum_i (x^i)^2} \geq R_0 , \qquad T_0 - R_0 + r \leq t < \infty , \tag{5.2.1}$$

such that the metric g satisfies there

$$|g_{\mu\nu} - \eta_{\mu\nu}| \leq C_1 r^{-\alpha} \leq C_2 , \quad \alpha > 0 , \tag{5.2.2}$$

for some positive constants C_1, C_2, α; clearly C_2 can be chosen to be less than or equal to $C_1 R_0^{-\alpha}$. Making R_0 larger one can thus make C_2 as small as desired, e.g.

$$C_2 = 10^{-2} , \tag{5.2.3}$$

which is a convenient number in dimension $3+1$ to guarantee that objects algebraically constructed out of g (such as $g^{\mu\nu}$, $\sqrt{\det g}$) are well controlled; (5.2.3) is certainly not optimal, and any other number suitable for the purposes at hand would do. To be able to prove theorems about such spacetimes one would need to impose some further, perhaps not necessarily uniform, decay conditions on a finite number of derivatives of g; there are various possibilities here, but we shall ignore this for the moment. Then one can define the *exterior region* $\mathscr{M}_{\rm ext}$, the *black hole region* $\mathscr{B}$ and the *future event horizon* $\mathscr{E}$ as

$$\mathscr{M}_{\rm ext} := \cup_{\tau \geq T_0} J^-(\mathcal{S}_{\tau,R_0}) = J^-(\cup_{\tau \geq T_0} \mathcal{S}_{\tau,R_0}) , \tag{5.2.4}$$

$$\mathcal{B} := \mathscr{M} \setminus \mathscr{M}_{\rm ext} , \quad \mathscr{E} := \partial \mathcal{B} , \tag{5.2.5}$$

where

$$\mathcal{S}_{\tau,R_0} := \{t = \tau, r = R_0\} . \tag{5.2.6}$$

We will refer to the definition (5.2.1)-(5.2.6) as that of a *naive* black hole.

In the setup of Equations (5.2.1)-(5.2.6) an arbitrarily chosen R_0 has been used; for this definition to make sense $\mathcal{B}$ so defined should *not* depend upon this choice. This is indeed the case, as can be seen as follows:

PROPOSITION 5.2.1 *Let* $\mathscr{O}_a \subset \mathbb{R}^3 \setminus B(0, R_0)$, $a = 1, 2$, *and let* $\mathscr{U}_a \subset \mathscr{M}$ *be of the form* $\{(t \geq T_0 - R_0 + r(\vec{x}), \vec{x})\ , \vec{x} \in \mathscr{O}_a\}$ *in the coordinate system of (5.2.2). Then*

$$I^-(\mathscr{U}_1) = J^-(\mathscr{U}_1) = I^-(\mathscr{U}_2) = J^-(\mathscr{U}_2)\ .$$

PROOF: If Γ is a future directed causal path from $p \in \mathscr{M}$ to $q = (t, \vec{x}) \in \mathscr{U}_1$, then the path obtained by concatenating Γ with the path $[0, 1] \ni s \to (t(s) := t + s, \vec{x}(s) := \vec{x})$ is a causal path which is not a null geodesic, hence can be deformed to a timelike path from p to $(t + 1, \vec{x}) \in \mathscr{U}_1$. It follows that $I^-(\mathscr{U}_1) = J^-(\mathscr{U}_1)$; clearly the same holds for $\mathscr{U}_2$. Next, let $\vec{x}_a \in \mathscr{O}_a$, and let $\gamma : [0, 1] \to \mathbb{R}^3 \setminus B(0, R_0)$ be any differentiable path such that $\gamma(0) = \vec{x}_1$ and $\gamma(1) = \vec{x}_2$. Then for any $t_0 \geq T_0 - R_0 + r(\vec{x}_1)$ the causal curve $[0, 1] \ni s \to \Gamma(s) = (t := Cs + t_0, \vec{x}(s) := \gamma(s))$ will be causal for the metric g by (5.2.2) if the constant C is chosen large enough, with a similar result holding when $\vec{x}_1$ is interchanged with $\vec{x}_2$. The equality $I^-(\mathscr{U}_1) = I^-(\mathscr{U}_2)$ easily follows from this observation. □

Summarizing, Proposition 5.2.1 shows that there are many possible equivalent definitions of $\mathscr{M}_{\rm ext}$: in (5.2.4) one can replace $J^-(\mathcal{S}_{\tau,R_0})$ by $J^-(\mathcal{S}_{\tau,R_1})$ for any $R_1 \geq R_0$, but also simply by $J^-((t + \tau, q))$, for any $p = (t, q) \in \mathscr{M}$ which belongs to the region covered by the coordinate system (t, x^i).

The following remarks concerning the definition Equations (5.2.4)-(5.2.5) are in order:

- For vacuum, stationary, asymptotically flat spacetimes the definition is equivalent to the usual one with $\mathscr{I}$ [96, Footnote 7, p. 572]; here the results of [102, 104] are used. However, one does not expect the existence of a smooth $\mathscr{I}^+$ to follow from (5.2.1)-(5.2.2) in general.

- Suppose that $\mathscr{M}$ admits a conformal completion, and that $\mathscr{I}$ is *semi-complete to the future.* Then for any finite interval $[T_0, T_1]$ there exists $R_0(T_0, T_1)$ and a coordinate system satisfying (5.2.2) and covering a set $r \geq R_0(T_0, T_1)$, $T_0 - R_0 \leq t \leq T_1 - R_0$. This follows from the Tamburino-Winicour construction of Bondi coordinates (u, r, θ, φ) near $\mathscr{I}^+$ [262], followed by the introduction of the usual Minkowskian coordinates $t = u + r$, $x = r \sin\theta \cos\varphi$, *etc.* The problem is that $R(T_1, T_2)$ could shrink to zero as T_2 goes to infinity. Thus, when $\mathscr{I}^+$ exists, conditions Equations (5.2.1)-(5.2.2) are *uniformity* conditions on $\mathscr{I}^+$ to the future: the metric remains uniformly controlled on a uniform neighborhood of $\mathscr{I}^+$ as the retarded time goes to infinity.

- It should not be too difficult to check whether or not the future geodesically complete spacetimes of Friedrich [119, 123], Christodoulou and Klainerman [54], Lindblad and Rodnianski [**?**], Hintz and Vasy [151], or the small data black holes of [100, 170], as well as the Robinson-Trautman black-holes discussed in Chapter 6 admit coordinate systems satisfying (5.2.1)-(5.2.2).

It is not clear if asymptotically flat spacetimes in which no such control is available do exist at all; in fact, it is tempting to formulate the following version of the *(weak) cosmic censorship* conjecture:

> The maximal globally hyperbolic development of generic[5], asymptotically flat, vacuum initial data contains a region with coordinates satisfying (5.2.1)-(5.2.2).

Whatever the status of this conjecture, one can hardly envisage numerical simulations leading to the calculation of an essential fraction of the total energy radiated away in spacetimes in which some uniformity conditions do not hold.

5.2.2 Quasi-local black holes

As already argued, the naive approach of the previous section should be more convenient for numerical simulations of black hole spacetimes, as compared to the usual one based on Scri. It appears to be even more convenient to have a framework in which all the issues are localized in space; we wish to suggest such a framework here. When numerically modeling an asymptotically flat spacetime, whether in a conformal or a direct approach, a typical numerical grid will contain large spheres S(R) on which the metric is nearly flat, so that an inequality such as (5.2.2)-(5.2.3) will hold in a neighborhood of S(R). On slices $t = \text{const}$ the condition (5.2.2) is usually complemented with a fall-off condition on the derivatives of the metric

$$|\partial_\sigma g_{\mu\nu}| \leq C r^{-\alpha-1} \; . \tag{5.2.7}$$

However, condition (5.2.7) is inadequate in the radiation regime, where retarded time derivatives of the metric are not expected to fall-off faster than r^{-1}. It turns out that there is a condition on derivatives of the metric in null directions which is fulfilled at large distance both in spacelike and in null directions: Let K_a, $a = 1, 2$, be null future pointing vector fields on $S(R)$ orthogonal to $S(R)$, with K_1 inwards pointing and K_2 – outwards pointing; these vector fields are unique up to scaling. Let χ_a denote the associated null second fundamental forms defined as

$$\forall\ X, Y \in TS(R) \quad \chi_a(X, Y) := g(\nabla_X K_a, Y) \; . \tag{5.2.8}$$

It can be checked, e.g. using the asymptotic expansions for the connection coefficients near $\mathscr{I}^+$ from [80, Appendix C], that χ_1 is negative definite and χ_2 is positive definite for Bondi spheres $S(R)$ sufficiently close to $\mathscr{I}^+$; similarly for $\mathscr{I}^-$. This property is not affected by the rescaling freedom at hand. Following G. Galloway [126], a two-dimensional spacelike submanifold of a four-dimensional spacetime will be called *weakly null convex* if χ_1 is semi-positive definite, with the trace of χ_2 negative.[6] The null convexity condition is easily

[5]The examples constructed by Christodoulou [53] with spherically symmetric gravitating scalar fields suggest that the genericity condition is unavoidable, though no corresponding vacuum examples are known.

[6]Galloway defines *null convexity* through the requirement of positive definiteness of χ_1 and negative definiteness of χ_2. However, he points out himself [126, p. 1472] that the weak null convexity as defined above suffices for his arguments to go through.

verified for sufficiently large spheres in a region asymptotically flat in the sense of (5.2.7). It does also hold for large spheres in a large class of spacetimes with negative cosmological constant. The null convexity condition is then the condition which we propose as a starting point to defining "quasi-local" black holes and horizons. The point is that several of the usual properties of black holes carry over to the weakly null convex setting. In retrospect, the situation can be summarized as follows: the usual theory of Scri based black holes exploits the existence of conjugate points on appropriate null geodesics whenever those are complete to the future; this completeness is guaranteed by the fact that the conformal factor goes to zero at the conformal boundary at an appropriate rate. Galloway's discovery in [126] is that weak null convexity of large spheres near Scri provides a second, in principle completely independent, mechanism to produce the needed focusing behaviour.

Throughout this section we will consider a globally hyperbolic spacetime $(\mathscr{M}, g)$ with time function t. Let $\mathscr{T} \subset \mathscr{M}$ be a finite union of connected timelike hypersurfaces $\mathscr{T}_\alpha$ in $\mathscr{M}$. We set

$$\mathscr{S}_\tau := \{t = \tau\}\ , \quad \mathscr{T}(\tau) := \mathscr{T} \cap \mathscr{S}_\tau\ , \quad \mathscr{T}_\alpha(\tau) := \mathscr{T}_\alpha \cap \mathscr{S}_\tau\ . \tag{5.2.9}$$

For further purposes anything that happens on the exterior side of $\mathscr{T}$ is completely irrelevant, so it is convenient to think of $\mathscr{T}$ as a boundary of $\mathscr{M}$; global hyperbolicity should then be understood in the sense that $(\overline{\mathscr{M}} := \mathscr{M} \cup \mathscr{T}, g)$ is strongly causal, and that $J^+(p; \overline{\mathscr{M}}) \cap J^-(q; \overline{\mathscr{M}})$ is compact in $\overline{\mathscr{M}}$ for all $p, q \in \overline{\mathscr{M}}$.

One can also think of each $\mathscr{T}_\alpha$ as a family of observers.

Recall that the null convergence condition is the requirement that

$$\mathrm{Ric}(X, X) \geq 0 \quad \text{for all null vectors } X \in TM\ . \tag{5.2.10}$$

We have the following *topological censorship* theorem for weakly null convex timelike boundaries:

THEOREM 5.2.2 (Galloway [126]) *Suppose that a globally hyperbolic spacetime $(\overline{\mathscr{M}}, g)$ satisfying the null convergence condition (5.2.10) has a timelike boundary $\mathscr{T} = \cup_{\alpha=1}^I \mathscr{T}_\alpha$ and a time function t such that the level sets of t are Cauchy surfaces, with each section $\mathscr{T}(\tau)$ of $\mathscr{T}$ being null convex. Then distinct $\mathscr{T}_\alpha$'s cannot communicate with each other:*

$$\alpha \neq \beta \qquad J^+(\mathscr{T}_\alpha) \cap J^-(\mathscr{T}_\beta) = \emptyset\ .$$

As is well known, topological censorship implies constraints on the topology:

THEOREM 5.2.3 (Galloway [126]) *Under the hypotheses of Theorem 5.2.2 suppose further that the cross-sections $\mathscr{T}_\alpha(\tau)$ of $\mathscr{T}_\alpha$ have spherical topology.*[7] *Then the* α-domain of outer communication

$$\langle\langle \mathscr{T}_\alpha \rangle\rangle := J^+(\mathscr{T}_\alpha) \cap J^-(\mathscr{T}_\alpha) \tag{5.2.11}$$

is simply connected.

[7]The reader is referred to [128] and references therein for results without the hypothesis of spherical topology. The results there, presented in a Scri setting, generalize immediately to the weakly null convex one.

It follows in particular from Theorem 5.2.3 that $\overline{\mathscr{M}}$ can be replaced by a subset thereof such that $\mathscr{T}$ is connected in the new spacetime, with all essential properties relevant for the discussion in the remainder of this section being unaffected by that replacement. We shall not do that, to avoid a lengthy discussion of which properties are relevant and which are not, but the reader should keep in mind that the hypothesis of connectedness of $\mathscr{T}$ can indeed be done without any loss of generality for most purposes.

We define the *quasi-local black hole region* $\mathscr{B}_{\mathscr{T}_\alpha}$ and the *quasi-local event horizon* $\mathscr{E}_{\mathscr{T}_\alpha}$ associated with the hypersurface $\mathscr{T}_\alpha$ by

$$\mathscr{B}_{\mathscr{T}_\alpha} := \mathscr{M} \setminus J^-(\mathscr{T}_\alpha) , \qquad \mathscr{E}_{\mathscr{T}_\alpha} := \partial \mathscr{B}_{\mathscr{T}_\alpha} . \tag{5.2.12}$$

If $\mathscr{T}$ is the hypersurface $\cup_{\tau \geq T_0} \mathcal{S}_{\tau,R_0}$ of Section 5.2.1 then the resulting black hole region coincides with that defined in (5.2.5), hence does not depend upon the choice of R_0 by Proposition 5.2.1; however, $\mathscr{B}_{\mathscr{T}_\alpha}$ might depend upon the chosen family of observers $\mathscr{T}_\alpha$ in general. It is certainly necessary to impose some further conditions on $\mathscr{T}$ to reduce this dependence. A possible condition, suggested by the geometry of the large coordinate spheres considered in the previous section, could be that the light-cones of the induced metric on $\mathscr{T}$ are uniformly controlled both from outside and inside by those of two static, future complete reference metrics on $\mathscr{T}$. However, neither the results above, nor the results that follow, do require that condition.

The Scri-equivalents of Theorem 5.2.3 [35, 65, 74, 96, 125, 128, 129, 163] allow one to control the topology of "good" sections of the horizon, and for the standard stationary black-holes this does lead to the usual $S^2 \times \mathbb{R}$ topology of the horizon [96, 147]. In particular, in stationary, asymptotically flat, appropriately regular spacetimes the intersection of a partial Cauchy hypersurface with an event horizon will necessary be a finite union of spheres. In general spacetimes such intersections do not even need to be manifolds: for example, in the usual spherically symmetric collapsing star the intersection of the event horizon with level sets of a time function will be a point at the time of appearance of the event horizon. We refer the reader to [73, Section 3] for other such examples, including one in which the topology of sections of horizon changes type from toroidal to spherical as time evolves. This behaviour can be traced back to the existence of past end points of the generators of the horizon. Nevertheless, some sections of the horizon have controlled topology – for instance, we have the following:

Theorem 5.2.4 *Under the hypotheses of Theorem 5.2.2, consider a connected component $\mathscr{T}_\alpha$ of $\mathscr{T}$ such that $\mathscr{E}_{\mathscr{T}_\alpha} \neq \emptyset$. Let*

$$\mathscr{C}_\alpha(\tau) := \partial J^+(\mathscr{T}_\alpha(\tau)) .$$

If $\mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$ is a topological manifold, then each connected component thereof has spherical topology.

Proof: Consider the open subset $\mathscr{M}_\tau$ of $\overline{\mathscr{M}}$ defined as

$$\mathscr{M}_\tau := I^+(\mathscr{C}_\alpha(\tau); \overline{\mathscr{M}}) \cap I^-(\mathscr{T}_\alpha; \overline{\mathscr{M}}) \subset \langle\langle \mathscr{T}_\alpha \rangle\rangle .$$

We claim that $(\mathscr{M}_\tau, g|_{\mathscr{M}_\tau})$ is globally hyperbolic: indeed, let $p, q \in \mathscr{M}_\tau$; global hyperbolicity of $\overline{\mathscr{M}}$ shows that $J^-(p;\overline{\mathscr{M}}) \cap J^+(q;\overline{\mathscr{M}})$ is a compact subset of $\overline{\mathscr{M}}$, which is easily seen to be included in $\mathscr{M}_\tau$. It follows that $J^-(p;\mathscr{M}_\tau) \cap J^+(q;\mathscr{M}_\tau)$ is compact, as desired. By the usual decomposition we thus have

$$\mathscr{M}_\tau \approx \mathbb{R} \times \mathcal{S} \ ,$$

where $\mathcal{S}$ is a Cauchy hypersurface for $\mathscr{M}_\tau$. Applying Theorem 5.2.3 to the globally hyperbolic spacetime $\mathscr{M}_\tau$ (which has a weakly null convex boundary $\mathscr{T}_\alpha \cap \{t > \tau\}$) one finds that $\mathscr{M}_\tau$ is simply connected, and thus so is $\mathcal{S}$. Since $\mathscr{C}_\alpha(\tau)$ and $\mathscr{E}_\alpha$ are null hypersurfaces in $\mathscr{M}$, it is easily seen that the closure in $\overline{\mathscr{M}}$ of the Cauchy surface $\{0\} \times \mathcal{S}$ intersects $\mathscr{E}_\alpha$ precisely at $\mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$. It follows that $\mathcal{S}$ is a compact, simply connected, three dimensional topological manifold with boundary, and a classical result [148, Lemma 4.9] shows that each connected component of $\partial\mathcal{S}$ is a sphere. The result follows now from $\partial\mathcal{S} \approx \mathscr{C}_\alpha(\tau) \cap \mathscr{E}_{\mathscr{T}_\alpha}$. □

Yet another class of "good sections" of $\mathscr{E}_{\mathscr{T}}$ can be characterized[8] as follows: suppose that $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ is a submanifold with boundary of $\mathscr{M}$ which is, moreover, a retract of $\langle\langle \mathscr{T}_\alpha \rangle\rangle$. Then $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ is simply connected by Theorem 5.2.3, and spherical topology of all boundary components of $\langle\langle \mathscr{T}_\alpha \rangle\rangle \cap \mathscr{S}_\tau$ follows again from [148, Lemma 4.9]. It is not clear whether there always exist time functions t such that the retract condition is satisfied; similarly it is not clear that there always exist τ's for which the conditions of Theorem 5.2.4 are met for metrics which are not stationary (one would actually want "a lot of τ's"). It would be of interest to understand this better.

We have an area theorem for $\mathscr{E}_{\mathscr{T}}$:

THEOREM 5.2.5 *Under the hypotheses of Theorem 5.2.2, suppose further that $\mathscr{E}_{\mathscr{T}} \neq \emptyset$. Let $\mathscr{S}_a$, $a = 1, 2$ be two achronal spacelike embedded hypersurfaces of C^2 differentiability class, set $S_a = \mathscr{S}_a \cap \mathscr{E}_{\mathscr{T}}$. Then:*

1. *The area of S_a is well defined.*
2. *If*

$$S_1 \subset J^-(S_2) \ ,$$

then the area of S_2 is larger than or equal to that of S_1. (Moreover, this is true even if the area of S_1 is counted with multiplicity[9] of generators provided that $S_1 \cap S_2 = \emptyset$.)

We note that point 1 is less trivial as it appears, because horizons can be rather rough sets, and it requires a certain amount of work to establish that claim.

PROOF: The result is obtained by a mixture of methods of [73] and of [126], and proceeds by contradiction: assume that the Alexandrov divergence $\theta_{\mathcal{A}l}$ of $\mathscr{E}_{\mathscr{T}}$ is negative, and consider the $S_{\epsilon,\eta,\delta}$ deformation of the horizon as constructed

[8] I am grateful to G. Galloway for useful discussions concerning this question, as well as many other points presented in this section.

[9] See [73] for details.

in Proposition 4.1 of [73], with parameters chosen so that $\theta_{\epsilon,\eta,\delta} < 0$. Global hyperbolicity implies the existence of an achronal null geodesic from $S_{\epsilon,\eta,\delta}$ to some cut $\mathscr{T}(\tau)$ of $\mathscr{T}$. The geodesic can further be chosen to be "extremal", in the sense that it meets $\mathscr{T}(t)$ for the smallest possible value of t among all generators of the boundary of $J^+(S_{\epsilon,\eta,\delta})$ meeting $\mathscr{T}$. The argument of the proof of Theorem 1 of [126] shows that this is incompatible with the null energy condition and with weak null convexity of $\mathscr{T}(\tau)$. It follows that $\theta_{\mathcal{A}l} \geq 0$, and the result follows from [73, Proposition 3.3 and Theorem 6.1]. $\square$

It immediately follows from the proof above that, under the hypotheses of Theorem 5.2.2, the occurrence of twice differentiable future trapped (compact) surfaces implies the presence of a black hole region. The same result holds for semi-convex compact surfaces which are trapped in an Alexandrov sense. It is, however, not known if the existence of marginally trapped surfaces – whether defined in a classical, or Alexandrov, or a viscosity sense – does signal the occurrence of black hole; it would be of interest to settle that.

In summary, we have shown that the quasi-local black holes, defined using weakly null convex timelike hypersurfaces, or boundaries, possess several properties usually associated with the Scri-based black holes, without the associated problems. We believe they provide a reasonable alternative, well suited for numerical calculations.

Chapter 6

Dynamical black holes: the Robinson-Trautman metrics

All black-hole metrics seen so far were stationary. This is essentially due to the fact that explicit non-stationary metrics are hard to come by. The closest one can come to an explicit time-dependent black hole metric is provided by the Robinson-Trautman family metrics, which are explicit up to one function satisfying a parabolic evolution equation. The aim of this chapter is to present some properties of those metrics.

6.1 Robinson–Trautman spacetimes.

The Robinson–Trautman (RT) metrics are vacuum metrics which can be viewed as evolving from data prescribed on a single null hypersurface.

From a physical point of view, the RT metrics provide examples of isolated gravitationally radiating systems. In fact, these metrics were hailed to be the first exact nonlinear solutions describing such a situation. Their discovery [247] was a breakthrough in the conceptual understanding of gravitational radiation in Einstein's theory.

The RT metrics were the only example of vacuum dynamical black holes without any symmetries and with exhaustively described global structure until the construction, in 2013 [99], of a large class of such spacetimes using "scattering data" at the horizon and at future null infinity. Further dynamical black holes have been meanwhile constructed in 2018 in [100, 170], by evolution of small perturbations of Schwarzschild initial data. See also [83] for asymptotically many-black-hole dynamical vacuum spacetimes with "a piece of $\mathscr{I}$", and [166] for a class of vacuum multi-black-holes with a positive cosmological constant.

There are several interesting features exhibited by the RT metrics: First, and rather unexpectedly, in this class of metrics the Einstein equations reduce to a single parabolic fourth order equation. Next, the evolution is unique within the class, in spite of a "naked singularity" at $r = 0$. Last but not least, they possess remarkable extendibility properties.

By definition, the Robinson–Trautman spacetimes can be foliated by a *null,* *hypersurface-orthogonal, shear-free, expanding* geodesic congruence. It has been shown by Robinson and Trautman [243] that in such a spacetime there always exists a coordinate system in which the metric takes the form

$$ {}^4g = -\Phi\, du^2 - 2du\, dr + r^2 e^{2\lambda} \underbrace{\mathring{g}_{ab}(x^c)\, dx^a\, dx^b}_{=:\mathring{g}}, \quad \lambda = \lambda(u, x^a)\,, \quad (6.1.1) $$

$$ \Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r}\,, \quad R = R(g_{ab}) \equiv R(e^{2\lambda}\mathring{g}_{ab})\,, \quad (6.1.2) $$

where the x^a's are local coordinates on the two-dimensional smooth Riemannian manifold $({}^2M, \mathring{g})$, $m \neq 0$ is a constant which is related to the total Trautman-Bondi mass of the metric, and R is the Ricci scalar of the metric $g := e^{2\lambda}\mathring{g}$. In writing (6.1.1)-(6.1.2) we have ignored those spacetimes which admit a congruence as above and where the parameter m vanishes.

The Einstein equations for a metric of the form (6.1.1) reduce to a single equation

$$ \partial_u g_{ab} = \frac{1}{12m}\Delta_g R\, g_{ab} \quad \Longleftrightarrow \quad \partial_u \lambda = \frac{1}{24m} e^{-2\lambda}\Delta_{\mathring{g}}(e^{-2\lambda}(\mathring{R} - 2\Delta_{\mathring{g}}\lambda))\,, \quad (6.1.3) $$

where Δ_g is the Laplace operator of the two-dimensional metric $g = g_{ab}dx^a dx^b$, and $\mathring{R}$ is the Ricci scalar of the metric $\mathring{g}$.

Equation (6.1.3) will be referred to as the RT equation. It is first-order in the "time" u, fourth-order in the space-variables x^a, and belongs to the family of parabolic equations. The Cauchy data for (6.1.3) consist of a function $\lambda_0(x^a) \equiv \lambda(u = u_0, x^a)$, which is equivalent to prescribing the metric $g_{\mu\nu}$ of the form (6.1.1) on a null hypersurface $\{u = u_0,\ r \in (0, \infty)\} \times {}^2M$. Without loss of generality, translating u if necessary, we can assume that $u_0 = 0$.

Note that the initial data hypersurface asymptotes to a curvature singularity at $r = 0$, with the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging as r^{-6} when $r = 0$ is approached. This is a "white hole singularity", familiar to all known stationary black hole spaces-times.

The RT equation (6.1.3) has been considered in a completely different context by Calabi [38].

The function $\lambda \equiv 0$ solves (6.1.3) when $\mathring{g}$ is the unit round metric on the sphere. The metric (6.1.1) is then the Schwarzschild metric in retarded Eddington-Finkelstein coordinates.

It follows from the theory of parabolic equations that for $m < 0$ the evolution problem for (6.1.3) is locally well posed backwards in u, while for $m > 0$ the RT equation can be locally solved forwards in u. Redefining u to $-u$ transforms (6.1.3) with $m < 0$, $u \leq 0$ to the same equation with a new mass parameter $-m > 0$ and with $u \geq 0$. Thus, when discussing (6.1.3) it suffices to assume $m > 0$. On the other hand, the global properties of the associated spacetimes will be different, and will need separate discussion.

Note that solutions of typical parabolic equations, including (6.1.3), immediately become analytic. This implies that for smooth but *not analytic* initial functions λ_0, the equation will *not* be solvable backwards in u when $m > 0$, or forwards in u when $m < 0$.

In [58, 59, 91] the following has been proved:

1. When $m > 0$ solutions of (6.1.3) with, say smooth, initial data at $u = 0$ exist for all $u \geq 0$. The proof consists in showing that all Sobolev norms of the solution remain finite during the evolution. The first key to this is the monotonicity of the Trautman-Bondi mass, which for RT metrics equals [257]
$$m_{\mathrm{TB}} = \frac{m}{4\pi}\int_{S^2} e^{3\lambda} d\mu_{\mathring{g}}\,. \tag{6.1.4}$$
The second is the monotonicity property of
$$\int_{{}^2M} (R - R_0)^2 \tag{6.1.5}$$
discovered by Calabi [38] and, independently, by Lukács, Perjes, Porter and Sebestyén [192].

2. Let $m > 0$. There exists a strictly increasing sequence of real numbers $\lambda_i > 0$, integers n_i with $n_1 = 0$, and functions $\varphi_{i,j} \in C^\infty({}^2M)$, $0 \leq j \leq n_i$, such that, possibly after performing a conformal transformation of $\mathring{g}$, solutions of (6.1.3) have a full asymptotic expansion of the form
$$\lambda(u, x^a) = \sum_{i\geq 1\,,\ 0\leq j\leq n_i} \varphi_{i,j}(x^a) u^j e^{-\lambda_i u/m}\,, \tag{6.1.6}$$
when u tends to infinity. The result is obtained by a delicate asymptotic analysis of solutions of the RT equation.

The decay exponents λ_i and the n_i's are determined by the spectrum of $\Delta_{\mathring{g}}$. For example, if $({}^2M, \mathring{g})$ is a round two sphere, we have [59]
$$\lambda_i = 2i\,,\ i \in \mathbb{N}\,,\ \text{with } n_1 = \ldots = n_{14} = 0\,,\ n_{15} = 1\,. \tag{6.1.7}$$

REMARK 6.1.1 The first global existence result for the RT equation has been obtained by Rendall [244] for a restricted class of near-Schwarzschildian initial data. Global existence and convergence to a round metric for all smooth initial data has been established in [58]. There the uniformization theorem for compact two-dimensional manifolds has been assumed. An alternative proof of global existence, which establishes the uniformization theorem as a by-product, has been given by Struwe [260]. □

The RT metrics all possess a smooth conformal boundary *à la Penrose* at "$r = \infty$". To see this, one can replace r by a new coordinate $x = 1/r$, which brings the metric (6.1.1) to the form
$${}^4g = x^{-2}\left(-\left(\frac{Rx^2}{2} + \frac{x\Delta_g R}{12m} - 2mx^3\right) du^2 + 2du\,dx + e^{2\lambda}\mathring{g}\right)\,, \tag{6.1.8}$$
so that the metric ${}^4g$ multiplied by a conformal factor x^2 smoothly extends to $\{x = 0\}$.

In what follows we shall take $({}^2M, \mathring{g})$ to be a two dimensional sphere equipped with the unit round metric. See [59] for a discussion of other topologies.

6.1.1 $m > 0$

Let us assume that $m > 0$. Following an observation of Schmidt reported in [265], the hypersurface "$u = \infty$" can be attached to the manifold $\{r \in (0,\infty)\,,\ u \in [0,\infty)\} \times {}^2M$ as a null boundary by introducing Kruskal–Szekeres-type coordinates $(\hat u, \hat v)$, defined in a way identical to the ones for the Schwarzschild metric:

$$\hat u = -\exp\left(-\frac{u}{4m}\right)\,, \quad \hat v = \exp\left(\frac{u+2r}{4m} + \ln\left(\frac{r}{2m} - 1\right)\right)\,. \tag{6.1.9}$$

This brings the metric to the form

$$\begin{aligned} {}^4g \;=\; & -\frac{32m^3\exp\left(-\frac{r}{2m}\right)}{r}d\hat u\, d\hat v + r^2 e^{2\lambda}\mathring{g} \\ & -16m^2\exp\left(\frac{u}{2m}\right)\left(\frac{R}{2} - 1 + \frac{r\Delta_g R}{12m}\right) d\hat u^2\,. \end{aligned} \tag{6.1.10}$$

Note that ${}^4g_{\hat u\hat u}$ vanishes when $\lambda \equiv 0$, and one recovers the Schwarzschild metric in Kruskal-Szekeres coordinates. Equations (6.1.6)-(6.1.7) imply that ${}^4g_{\hat u\hat u}$ decays as $e^{u/2m} \times e^{-2u/m} = \hat u^6$. Hence g approaches the Schwarzschild metric as $O(\hat u^6)$ when the null hypersurface

$$\mathcal{H}^+ := \{\hat u = 0\}$$

is approached. A projection diagram, as defined in Section 4.7, [87], with the ${}^2M$ factor projected out, can be found in Figure 6.1.1.

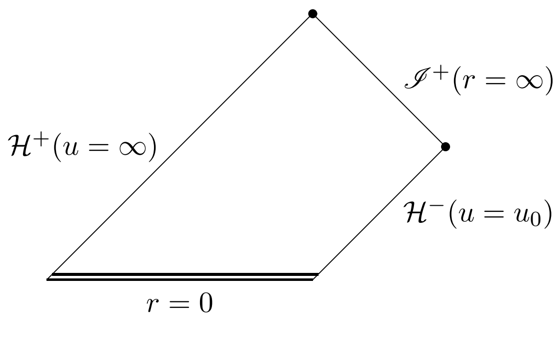

Figure 6.1.1: A projection diagram for RT metrics with $m > 0$.

In terms of $\hat u$ the expansion (6.1.6) becomes

$$\lambda(\hat u, x^a) = \sum_{i\geq 1\,,\ 0\leq j\leq n_i} \varphi_{i,j}(x^a)\,(-4m\log(|\hat u|))^j\,\hat u^{8i}\,, \tag{6.1.11}$$

which can be extended to $\hat u > 0$ as an even function of $\hat u$. This expansion carries over to similar expansions of R and $\Delta_g R$, and results in an asymptotic expansion of the form

$${}^4g_{\hat u\hat u}(\hat u, x^a) = \sum_{i\geq 1\,,\ 0\leq j\leq n_i} \psi_{i,j}(x^a)\,(\log|\hat u|)^j\,\hat u^{8i-2}\,, \tag{6.1.12}$$

for some functions ψ_{ij}. It follows from (6.1.2) that the even extension of ${}^4g_{\hat u\hat u}$ will be of C^{117}–differentiability class.

In fact, any two such even functions ${}^4g_{\hat u\hat u}$ can be continued into each other across $u = 0$ to a function of C^5–differentiability class. It follows that:

1. Any two RT metrics can be joined together as in Figure 6.1.2 to obtain a spacetime with a metric of C^5–differentiability class. In particular g can be glued to a Schwarzschild metric beyond $\mathcal{H}$, resulting in a C^5 metric.

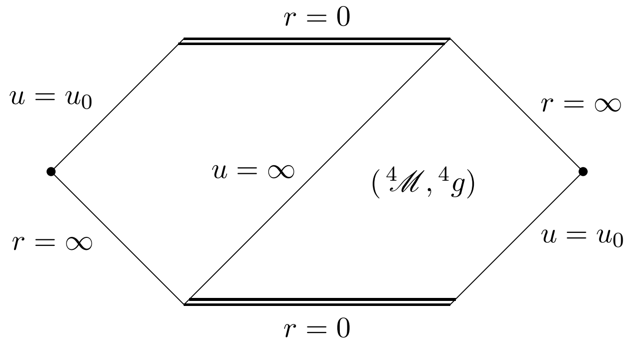

Figure 6.1.2: Vacuum RT extensions beyond $\mathcal{H}^+ = \{u = \infty\}$. Any two RT metrics with the same mass parameter m can be glued across the null hypersurface $\mathcal{H}^+$, leading to a metric of C^5–differentiability class.

2. It follows from (6.1.2) that g can be glued to itself in the C^{117}–differentiability class.

The vanishing, or not, of the expansion functions $\varphi_{i,j}$ in (6.1.6) with $j \geq 1$ turns out to play a key role for the smoothness of the metric at $\mathcal{H}$. Indeed, the first non-vanishing function $\varphi_{i,j}$ with $j \geq 1$ will lead to a $\psi_{i,j}\,(\ln|\hat u|)^j\,\hat u^{8i-2}$ term in the asymptotic expansion of ${}^4g_{\hat u\hat u}$. As a result, ${}^4g_{\hat u\hat u}$ will be extendable to an even function of $\hat u$ of C^{8i-3}-differentiability class, but not better. It is shown in [91] that

1. Generic $\lambda(0, x^a)$ close to zero lead to a solution with $\psi_{15,1} \neq 0$, resulting in metrics which are extendible across $\mathcal{H}$ in the C^{117}-differentiability class, but not C^{118}, in the coordinate system above.

2. There exists an infinite-dimensional family of non-generic initial functions $\lambda(0, x^a)$ for which $\psi_{15,1} \equiv 0$. An even extension of ${}^4g_{\hat u\hat u}$ across $\mathcal{H}$ results in a metric of C^{557}-differentiability class, but not C^{558}, in the coordinate system above.

The question arises, whether the above differentiability issues are related to a poor choice of coordinates. By analysing the behaviour of the derivatives of the Riemann tensor on geodesics approaching $\mathcal{H}$, one can show [91] that the metrics of point 1 above cannot be extended across $\mathcal{H}$ in the class of spacetimes with metrics of C^{123}-differentiability class. Similarly the metrics of point 2 cannot be extended across $\mathcal{H}$ in the class of spacetimes with metrics of C^{564}-differentiability class. One expects that the differentiability mismatches are not a real effect, but result from a non-optimal inextendibility criterion used. It would be of some interest to settle this issue.

Summarising, we have the following:

THEOREM 6.1.2 *Let $m > 0$. For any $\lambda_0 \in C^\infty(S^2)$ there exists a Robinson–Trautman spacetime $({}^4\mathscr{M}, {}^4g)$ with a "half-complete" $\mathscr{I}^+$, the global structure of which is shown in Figure 6.1.1. Moreover:*

1. *$({}^4\mathscr{M}, {}^4g)$ is smoothly extendible to the past through $\mathscr{H}^-$. If, however, λ_0 is* not *analytic, then* no *vacuum Robinson–Trautman extensions through $\mathscr{H}^-$ exist.*

2. *There exist infinitely many non-isometric vacuum Robinson–Trautman C^5 extensions*[1] *of $({}^4\mathscr{M}, {}^4g)$ through $\mathscr{H}^+$, which are obtained by gluing to $({}^4\mathscr{M}, {}^4g)$ any other positive mass Robinson–Trautman spacetime, as shown in Figure 6.1.2.*

3. *There exist infinitely many C^{117} vacuum RT extensions of $({}^4\mathscr{M}, {}^4g)$ through $\mathscr{H}^+$. One such extension is obtained by gluing a copy of $({}^4\mathscr{M}, {}^4g)$ to itself, as shown in Figure 6.1.2.*

4. *For any $6 \le k \le \infty$ there exists an open set $\mathcal{O}_k$ of Robinson–Trautman spacetimes, in a $C^k(S^2)$ topology on the set of the initial data functions λ_0, for which no C^{123} extensions beyond $\mathscr{H}^+$ exist, vacuum or otherwise. For any u_0 there exists an open ball $\mathcal{B}_k \subset C^k(S^2)$ around the initial data for the Schwarzschild metric, $\lambda_0 \equiv 0$, such that $\mathcal{O}_k \cap \mathcal{B}_k$ is dense in $\mathcal{B}_k$.*

The picture that emerges from Theorem 6.1.2 is the following: generic initial data lead to a spacetime which has no RT vacuum extension to the past of the initial surface, even though the metric can be smoothly extended (in the non–vacuum class); and generic data sufficiently close[2] to Schwarzschildian ones lead to a spacetime for which no smooth vacuum RT extensions exist beyond $\mathcal{H}^+$. This shows that considering smooth extensions across $\mathcal{H}^+$ leads to non–existence, while giving up the requirement of smoothness of extensions beyond $\mathcal{H}^+$ leads to non–uniqueness. It follows that global well-posedness of the general relativistic initial value problem completely fails in the class of positive mass Robinson–Trautman metrics.

REMARK 6.1.3 There are two striking differences between the global structure seen in Figure 6.1.2 and the usual Penrose diagram for Schwarzschild spacetime. The first is the lack of past null infinity, which we have seen to be unavoidable in the RT case. The second is the lack of the past event horizon, sections of which can be technically described as a *marginally past outer trapped surfaces*. The existence of such surfaces in RT spacetimes is a non-trivial property which has been established in [265]. □

[1]By this we mean that the metric can be C^5 extended beyond $\mathscr{H}^+$; the extension can actually be chosen to be of $C^{5,\alpha}$–differentiability class, for any $\alpha < 1$.

[2]It is rather clear from the results of [91] that generic RT spacetimes will not be smoothly extendible across $\mathcal{H}^+$, without any restrictions on the "size" of the initial data; but no rigorous proof is available.

6.1.2 $m < 0$

Unsurprisingly, and as already mentioned, the global structure of RT spacetimes turns out to be different when $m < 0$, which we assume now. As already noted, in this case we should take $u \leq 0$, in which case the expansion (6.1.6) again applies with $u \to -\infty$.

The existence of future null infinity as in (6.1.8) applies without further due, except that now the coordinate u belongs to $(-\infty, 0]$.

The new aspect is the possibility of attaching a conformal boundary at past null infinity, $\mathscr{I}^-$, which is carried out by first replacing u with a new coordinate v defined as [250]

$$v = u + 2r + 4m \ln\left(\left|\frac{r}{2m} - 1\right|\right) . \tag{6.1.13}$$

In the coordinate system (v, r, x^a) the metric becomes

$$\begin{aligned} {}^4g &= -\left(1 - \frac{2m}{r}\right) dv^2 + 2dv\, dr + r^2 e^{2\lambda} \mathring{g} \\ &\quad + \left(\frac{R}{2} - 1 + \frac{r}{12m}\Delta_g R\right)\left(dv - \frac{2dr}{1 - \frac{2m}{r}}\right)^2 . \end{aligned} \tag{6.1.14}$$

The last step is the usual replacement of r by $x = 1/r$:

$$\begin{aligned} {}^4g &= x^{-2}\Bigg[- x^2(1 - 2mx) dv^2 - 2dv\, dx + e^{2\lambda} \mathring{g} \\ &\quad + \left(\frac{R - 2}{2x^2} + \frac{12m\Delta_g R}{x^3}\right)\left(x^2 dv + \frac{2dx}{1 - 2mx}\right)^2 \Bigg] . \end{aligned} \tag{6.1.15}$$

One notices that all terms in the conformally rescaled metric $x^2 \times {}^4g$ extend smoothly to smooth functions of (v, x^a) at the conformal boundary $\{x = 0\}$ except possibly for

$$\left(\frac{R - 2}{2x^2} + \frac{12m\Delta_g R}{x^3}\right) \times \left(\frac{4dx^2}{(1 - 2mx)^2} + \frac{4x^2 dv\, dx}{1 - 2mx}\right) . \tag{6.1.16}$$

Now, from the definition of v we have

$$\begin{aligned} \exp\left(-\frac{2u}{m}\right) &= \left(\frac{r}{2m} - 1\right)^8 \exp\left(\frac{2v - 4r}{|m|}\right) \\ &= \left(\frac{1 - 2mx}{2mx}\right)^8 \exp\left(\frac{2v}{|m|}\right) \exp\left(-\frac{4}{|m|x}\right) . \end{aligned}$$

Using the fact that $\lambda = O(\exp(-2u/m))$, similarly for all angular derivatives of λ, we see that all three functions λ, $R - 2$ and $\Delta_g R$ decay to zero, as x approaches zero, faster than any negative power of x. In fact, the offending terms (6.1.16) extend smoothly by zero across $\{x = 0\}$. We conclude that the conformally rescaled metric $x^2 \times {}^4g$ smoothly extends to $\mathscr{I}^- := \{x = 0\}$.

Summarising, we have:

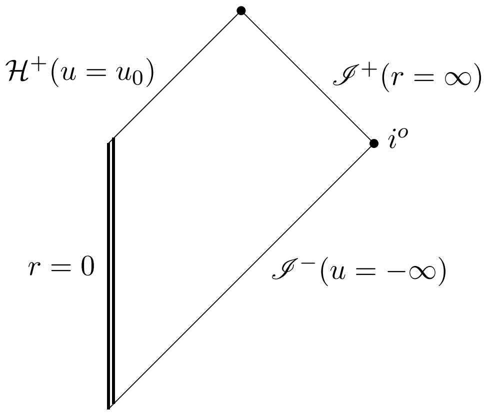

Figure 6.1.3: A projection diagram for RT metrics with $m < 0$.

THEOREM 6.1.4 *Let $m < 0$. For any $\lambda_0 \in C^\infty({}^2M)$ there exists a unique RT spacetime $({}^4\mathscr{M}, {}^4g)$ with a complete i^0 in the sense of [11], a complete $\mathscr{I}^-$, and "a piece of $\mathscr{I}^+$", as shown in Figure 6.1.3. Moreover*

1. *$({}^4\mathscr{M}, {}^4g)$ is smoothly extendible through $\mathscr{H}^+$, but*
2. *if λ_0 is* not *analytic, there exist* no *vacuum RT extensions through $\mathscr{H}^+$.*

The generic non-extendability of the metric through $\mathscr{H}^+$ in the vacuum RT class is rather surprising, and seems to be related to a similar non-extendability result for compact non–analytic Cauchy horizons in the polarized Gowdy class, *cf.* [79]. Since it may well be possible that there exist vacuum extensions which are not in the RT class, this result does not unambiguously demonstrate a failure of Einstein equations to propagate generic data forwards in u in such a situation; however, it certainly shows that the forward evolution of the metric via Einstein equations breaks down in the class of RT metrics with $m < 0$.

6.1.3 $\Lambda \neq 0$

So far we have assumed a vanishing cosmological constant. It turns out that there exists a straightforward generalisation of RT metrics to $\Lambda \neq 0$. The metric retains its form (6.1.1), with the function Φ of (6.1.2) taking instead the form

$$\Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r} - \frac{\Lambda}{3}r^2 . \tag{6.1.17}$$

We continue to assume that $m \neq 0$.

It turns out that the key equation (6.1.3) remains the same, thus λ tends to zero and Φ tends to the function

$$\mathring{\Phi} = \frac{\mathring{R}}{2} - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \tag{6.1.18}$$

as u approaches infinity. It follows from the generalised Birkhoff theorem 1.2.3 that these are the *Birmingham* metrics presented in Section 4.6. The relevant

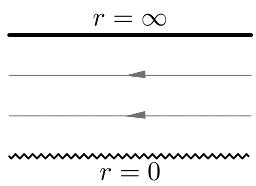

Figure 6.1.4: The causal diagram when $m < 0$, $\Lambda > 0$ and $\mathring{\Phi}$ has no zeros.

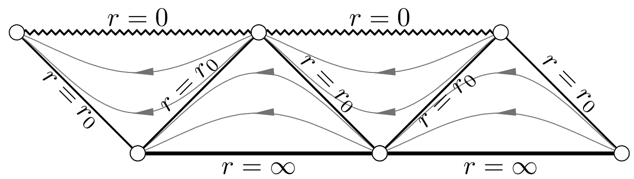

Figure 6.1.5: The causal diagram for Kottler metrics with $\Lambda > 0$, and $\mathring{\Phi} \leq 0$, with $\mathring{\Phi}$ vanishing precisely at r_0.

projection diagrams can be found there, for the convenience of the reader we repeat them in Figures 6.1.4-6.1.7.

The global structure of the spacetimes with $\Lambda \neq 0$ and $\lambda \not\equiv 0$ should be clear from the analysis of the case $\Lambda = 0$: One needs to cut one of the building blocs of Figures 6.1.4-6.1.7 with a line with a ± 45-degrees slope, corresponding to the initial data hypersurface $u_0 = 0$. This hypersurface should *not* coincide with one of the Killing horizons there, where $\mathring{\Phi}$ vanishes. The Killing horizons with the opposite slope in the diagrams should be ignored. Depending upon the sign of m, one can evolve to the future or to the past of the associated spacetime hypersurface until a conformal boundary at infinity or a Killing horizon $\mathring{\Phi}(r_0) = 0$ with the same slope is reached.

The metric will always be smoothly conformally extendable through the conformal boundaries at infinity.

As discussed in [30], the extendibility properties across the horizons which are approached as $m \times u$ tends to infinity will depend upon the surface gravity of the horizon and the spectrum of $\mathring{g}$. For simplicity we assume that

$$ {}^2M = S^2 \quad \Longleftrightarrow \quad \mathring{R} > 0 \,, $$

a similar analysis can be carried out for other topologies.

Consider, first, a zero $r = r_0$ of $\mathring{\Phi}$ such that

$$ c = \frac{\mathring{\Phi}'(r_0)}{2} > 0 \,. $$

Similarly to (6.1.9), introduce Kruskal–Szekeres-type coordinates $(\hat{u}, \hat{v})$ defined as

$$ \hat{u} = -e^{-cu} \,, \quad \hat{v} = e^{c(u+2F(r))} \,, \tag{6.1.19} $$

where

$$ F' = \frac{1}{\mathring{\Phi}} \,. \tag{6.1.20} $$

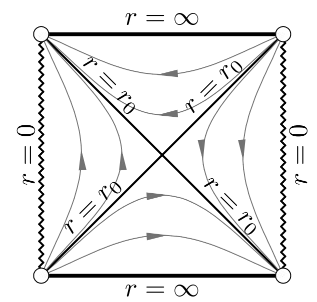

Figure 6.1.6: The causal diagram for Kottler metrics with $m < 0$, $\Lambda > 0$ $\mathring{R} \in \mathbb{R}$, or $m = 0$ and $\mathring{R} = 1$, with r_0 defined by the condition $\mathring{\Phi}(r_0) = 0$. The set $\{r = 0\}$ is a singularity unless the metric is the de Sitter metric (${}^2M = S^2$ and $m = 0$), or a suitable quotient thereof so that $\{r = 0\}$ corresponds to a center of (possibly local) rotational symmetry.

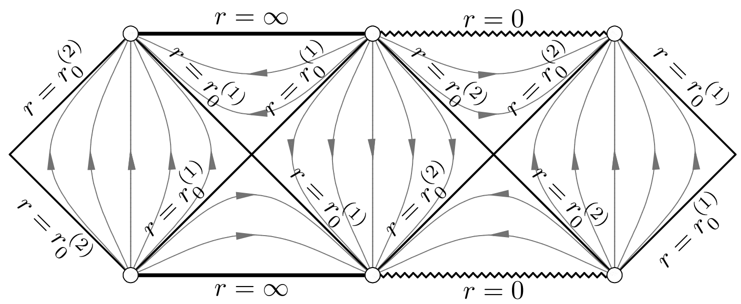

Figure 6.1.7: The causal diagram for Kottler metrics with $\Lambda > 0$ and exactly two first-order zeros of $\mathring{\Phi}$.

This brings the metric to the form

$$ {}^4g \quad = \quad -\frac{e^{-2cF(r)}\mathring{\Phi}}{c^2}\, d\hat{u}\, d\hat{v} + r^2 e^{2\lambda}\mathring{g} - \frac{e^{2cu}}{c^2}\Bigg(\underbrace{\frac{R-\mathring{R}}{2} + \frac{r\Delta_g R}{12m}}_{O(\exp(-2u/m))}\Bigg)\, d\hat{u}^2\,. \qquad (6.1.21) $$

It is elementary to show that ${}^4g_{\hat{u}\hat{v}}$ extends smoothly across $\{r = r_0\}$. Next we have

$$ {}^4g_{\hat{u}\hat{u}} \quad = \quad O\left(e^{2(c-\frac{1}{m})u}\right) = O\left(\hat{u}^{2(\frac{1}{mc}-1)}\right) \qquad (6.1.22) $$

which will extend continuously across a horizon $\{\hat{u} = 0\}$ provided that

$$ \frac{1}{mc} > 1 \qquad \Longleftrightarrow \qquad mF'(r_0) < 2\,. \qquad (6.1.23) $$

In fact when (6.1.23) holds, then for any $\epsilon > 0$ the extension to any other RT solution will be of $C^{\lfloor 2(\frac{1}{mc}-1)\rfloor-\epsilon}$ differentiability class.

When $\Lambda > 0$, the parameter $c = c(m, \Lambda)$ can be made as small as desired by making m approach from below the critical value

$$m_c = \frac{1}{3\sqrt{\Lambda}},$$

for which c vanishes. For $m > m_c$ the function $\mathring{\Phi}$ has no (real) zeros, and for $0 < m < m_c$ all zeros are simple.

It follows from Figure 6.1.8 that the extension through the black hole event horizon is at least of C^6-differentiability class, and becomes as differentiable as desired when the critical mass is approached.

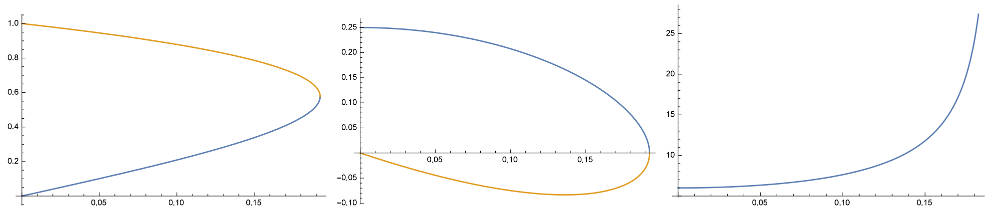

Figure 6.1.8: The value of the real positive zero of $\mathring{\Phi}$ (left plot), the product $m \times c(m, \Lambda)$ (middle plot), and the function $2/(m \times c(m, \Lambda)) - 2$ (which determines the differentiability class of the extension through the black hole event horizon; right plot) as functions of m, with $\mathring{R} = 2$ and $\Lambda = 3$.

The calculation above breaks down for degenerate horizons, where $m = m_c$, for which $c = 0$. In this case an extension across a degenerate horizon can be obtained by replacing u by a coordinate v defined as

$$v = u + 2F(r), \text{ with again } \frac{dF}{dr} = \frac{1}{\mathring{\Phi}}. \tag{6.1.24}$$

An explicit formula for F can be found, which is not very enlightening. Since $\mathring{\Phi}$ has a quadratic zero, we find that for r approaching r_0 we have, after choosing an integration constant appropriately,

$$u \approx v + \frac{1}{3(r - r_0)} \quad \Longrightarrow \quad \mathring{\Phi} \sim u^{-2} \text{ and } e^{-\frac{2u}{m}} \sim e^{-\frac{2}{3m(r-r_0)}}, \tag{6.1.25}$$

where $f \sim g$ is used to indicate that $|f/g|$ is bounded by a positive constant both from above and below over compact intervals of v.

Using $du = dv - 2dr/\mathring{\Phi}$ we find

$$^4g = -\Phi dv^2 + 2 \underbrace{\frac{2\Phi - \mathring{\Phi}}{\mathring{\Phi}}}_{1+O(\exp(-2u/m))} dv\, dr - 4 \underbrace{\frac{\Phi - \mathring{\Phi}}{\mathring{\Phi}^2}}_{O(u^4 \exp(-2u/m))} dr^2 + r^2 e^{2\lambda} \mathring{g}. \tag{6.1.26}$$

It easily follows that $^4g_{vr}$ can be *smoothly* extended by a constant function across $r = r_0$, and that $^4g_{rr}$ can be *again smoothly* extended by the constant function 0. We conclude that any RT metric with a degenerate horizon can be smoothly continued across the horizon to a Schwarzschild-de Sitter or Schwarzschild-anti de Sitter metric with the same mass parameter m, as first observed in [30].

INCIDENTALLY: Some results on higher-dimensional generalisations of RT metrics can be found in [235].

Part II

Background Material

Appendix A

Introduction to pseudo-Riemannian geometry

A.1 Manifolds

It is convenient to start with the definition of a manifold:

Definition A.1.1 *An n–dimensional manifold is a set M equipped with the following:*

1. *topology: a "connected Hausdorff paracompact topological space" (think of nicely looking subsets of $\mathbb{R}^{1+n}$, like spheres, hyperboloids, and such), together with*

2. *local charts: a collection of coordinate patches $(\mathscr{U}, x^i)$ covering M, where $\mathscr{U}$ is an open subset of M, with the functions $x^i : \mathscr{U} \to \mathbb{R}^n$ being continuous. One further requires that the maps*

$$M \supset \mathscr{U} \ni p \mapsto (x^1(p), \dots, x^n(p)) \in \mathscr{V} \subset \mathbb{R}^n$$

   are homeomorphisms.

3. *compatibility: given two overlapping coordinate patches, $(\mathscr{U}, x^i)$ and $(\widetilde{\mathscr{U}}, \tilde{x}^i)$, with corresponding sets $\mathscr{V}, \widetilde{\mathscr{V}} \subset \mathbb{R}^n$, the maps $\tilde{x}^j \mapsto x^i(\tilde{x}^j)$ are smooth diffeomorphisms wherever defined: this means that they are bijections differentiable as many times as one wishes, with*

$$\det\left[\frac{\partial x^i}{\partial \tilde{x}^j}\right] \text{ nowhere vanishing}.$$

Definition of differentiability: A function on M is smooth if it is smooth when expressed in terms of local coordinates. Similarly for tensors.

Examples:

1. $\mathbb{R}^n$ with the usual topology, one single global coordinate patch.

2. A sphere: use stereographic projection to obtain two overlapping coordinate systems (or use spherical angles, but then one must avoid borderline angles, so they don't cover the whole manifold!).

3. We will use several coordinate patches (in fact, five), to describe the Schwarzschild black hole, though one spherical coordinate system would suffice.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$, and define $N := f^{-1}(0)$. If ∇f has no zeros on N, then every connected component of N is a smooth $(n-1)$–dimensional manifold. This construction leads to a plethora of examples. For example, if $f = \sqrt{(x^1)^2 + \ldots + (x^n)^2} - R$, with $R > 0$, then N is a sphere of radius R.

In this context a useful example is provided by the function $f = t^2 - x^2$ on $\mathbb{R}^2$: its zero-level-set is the light-cone $t = \pm x$, which is a manifold except at the origin; note that $\nabla f = 0$ there, which shows that the criterion is sharp.

A.2 Scalar functions

Let M be an n-dimensional manifold. Since manifolds are defined using coordinate charts, we need to understand how things behave under coordinate changes. For instance, under a change of coordinates $x^i \to y^j(x^i)$, to a function $f(x)$ we can associate a new function $\bar{f}(y)$, using the rule

$$\bar{f}(y) = f(x(y)) \quad \Longleftrightarrow \quad f(x) = \bar{f}(y(x)) \,.$$

In general relativity it is a common abuse of notation to write the same symbol f for what we wrote $\bar{f}$, when we think that this is the same function but expressed in a different coordinate system. We then say that a real- or complex-valued f is a *scalar function* when, under a change of coordinates $x \to y(x)$, the function f transforms as $f \to f(x(y))$.

In this section, to make things clearer, we will write $\bar{f}$ for $f(x(y))$ even when f is a scalar, but this will almost never be done in the remainder of these notes. For example we will systematically use the same symbol $g_{\mu\nu}$ for the metric components, whatever the coordinate system used.

A.3 Vector fields

Physicists often think of vector fields in terms of coordinate systems: a vector field X is an object which in a coordinate system $\{x^i\}$ is represented by a collection of functions X^i. In a new coordinate system $\{y^j\}$ the field X is represented by a new set of functions:

$$X^i(x) \to X^j(y) := X^j(x(y)) \frac{\partial y^i}{\partial x^j}(x(y)) \,. \tag{A.3.1}$$

(The summation convention is used throughout, so that the index j has to be summed over.)

The notion of a vector field finds its roots in the notion of the tangent to a curve, say $s \to \gamma(s)$. If we use local coordinates to write $\gamma(s)$ as $(\gamma^1(s), \gamma^2(s), \ldots, \gamma^n(s))$, the tangent to that curve at the point $\gamma(s)$ is defined as the set of numbers

$$(\dot\gamma^1(s), \dot\gamma^2(s), \ldots, \dot\gamma^n(s)) \,.$$

Consider, then, a curve $\gamma(s)$ given in a coordinate system x^i and let us perform a change of coordinates $x^i \to y^j(x^i)$. In the new coordinates y^j the curve γ is represented by the functions $y^j(\gamma^i(s))$, with new tangent

$$\frac{dy^j}{ds}(y(\gamma(s))) = \frac{\partial y^j}{\partial x^i}(\gamma(s))\dot\gamma^i(s)\,.$$

This motivates (A.3.1).

In modern differential geometry a different approach is taken: one identifies vector fields with homogeneous first order differential operators acting on real valued functions $f : M \to \mathbb{R}$. In local coordinates $\{x^i\}$ a vector field X will be written as $X^i\partial_i$, where the X^i's are the "physicists's functions" just mentioned. This means that the action of X on functions is given by the formula

$$\boxed{X(f) := X^i\partial_i f} \tag{A.3.2}$$

(recall that ∂_i is the partial derivative with respect to the coordinate x^i). Conversely, given some abstract first order homogeneous derivative operator X, the (perhaps locally defined) functions X^i in (A.3.2) can be found by acting on the coordinate functions:

$$X(x^i) = X^i\,. \tag{A.3.3}$$

One justification for the differential operator approach is the fact that the tangent $\dot\gamma$ to a curve γ can be calculated — in a way independent of the coordinate system $\{x^i\}$ chosen to represent γ — using the equation

$$\dot\gamma(f) := \frac{d(f\circ\gamma)}{dt}\,.$$

Indeed, if γ is represented as $\gamma(t) = \{x^i = \gamma^i(t)\}$ within a coordinate patch, then we have

$$\frac{d(f\circ\gamma)(t)}{dt} = \frac{d(f(\gamma(t)))}{dt} = \frac{d\gamma^i(t)}{dt}(\partial_i f)(\gamma(t))\,,$$

recovering the previous coordinate formula $\dot\gamma = (d\gamma^i/dt)$.

An even better justification is that *the transformation rule (A.3.1) becomes implicit in the formalism.* Indeed, consider a (scalar) function f, so that the differential operator X acts on f by differentiation:

$$X(f)(x) := \sum_i X^i\frac{\partial f(x)}{\partial x^i}\,. \tag{A.3.4}$$

If we make a coordinate change so that

$$x^j = x^j(y^k) \quad\Longleftrightarrow\quad y^k = y^k(x^j)\,,$$

keeping in mind that

$$\bar f(y) = f(x(y)) \quad\Longleftrightarrow\quad f(x) = \bar f(y(x))\,,$$

then

$$\begin{aligned}
X(f)(x) &:= \sum_i X^i(x)\frac{\partial f(x)}{\partial x^i} \\
&= \sum_i X^i(x)\frac{\partial \bar f(y(x))}{\partial x^i} \\
&= \sum_{i,k} X^i(x)\frac{\partial \bar f(y(x))}{\partial y^k}\frac{\partial y^k}{\partial x^i}(x) \\
&= \sum_k \bar X^k(y(x))\frac{\partial \bar f(y(x))}{\partial y^k} \\
&= \left(\sum_k \bar X^k \frac{\partial \bar f}{\partial y^k}\right)(y(x)),
\end{aligned}$$

with $\bar X^k$ given by the right-hand side of (A.3.1). So

$\boxed{X(f) \text{ is a scalar iff the coefficients } X^i \text{ satisfy the transformation law of a vector.}}$

Exercice A.3.1 Check that this is a necessary and sufficient condition.

One often uses the middle formula in the above calculation in the form

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}\,. \tag{A.3.5}$$

Note that the tangent to the curve $s \to (s, x^2, x^3, \dots x^n)$, where $(x^2, x^3, \dots x^n)$ are constants, is identified with the differential operator

$$\partial_1 \equiv \frac{\partial}{\partial x^1}\,.$$

Similarly the tangent to the curve $s \to (x^1, s, x^3, \dots x^n)$, where $(x^1, x^3, \dots x^n)$ are constants, is identified with

$$\partial_2 \equiv \frac{\partial}{\partial x^2}\,,$$

etc. Thus, $\dot\gamma$ is identified with

$$\dot\gamma(s) = \dot\gamma^i\partial_i\,.$$

At any given point $p \in M$ the set of vectors forms a vector space, denoted by T_pM. The collection of all the tangent spaces is called the tangent bundle to M, denoted by TM.

A.3.1 Lie bracket

Vector fields can be added and multiplied by functions in the obvious way. Another useful operation is the *Lie bracket*, or *commutator*, defined as

$$\boxed{[X,Y](f) := X(Y(f)) - Y(X(f))}\,. \tag{A.3.6}$$

One needs to check that this does indeed define a new vector field: the simplest way is to use local coordinates,

$$\begin{aligned}
[X,Y](f) &= X^j\partial_j(Y^i\partial_i f) - Y^j\partial_j(X^i\partial_i f)\\
&= X^j(\partial_j(Y^i)\partial_i f + Y^i\partial_j\partial_i f) - Y^j(\partial_j(X^i)\partial_i f + X^i\partial_j\partial_i f)\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f + \underbrace{X^jY^i\partial_j\partial_i f - Y^jX^i\partial_j\partial_i f}_{=X^jY^i\underbrace{(\partial_j\partial_i f - \partial_i\partial_j f)}_{0}}\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f\,,
\end{aligned} \tag{A.3.7}$$

which is indeed a homogeneous first order differential operator. Here we have used the symmetry of the matrix of second derivatives of twice differentiable functions. We note that the last line of (A.3.7) also gives an explicit coordinate expression for the commutator of two differentiable vector fields.

The Lie bracket satisfies the *Jacobi identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\,.$$

Indeed, if we write $S_{X,Y,Z}$ for a cyclic sum, then

$$\begin{aligned}
&([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])(f) = S_{X,Y,Z}[X,[Y,Z]](f)\\
&\quad = S_{X,Y,Z}\left\{X([Y,Z](f)) - [Y,Z](X(f))\right\}\\
&\quad = S_{X,Y,Z}\left\{X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f)))\right\}\,.
\end{aligned}$$

The third term is a cyclic permutation of the first, and the fourth a cyclic permutation of the second, so the sum gives zero.

A.4 Covectors

Covectors are *maps from the space of vectors to functions which are linear under addition and multiplication by functions.*

The basic object is the *coordinate differential* dx^i, defined by its action on vectors as follows:

$$dx^i(X^j\partial_j) := X^i\,. \tag{A.4.1}$$

Equivalently,

$$dx^i(\partial_j) := \delta^i_j := \begin{cases} 1, & i=j;\\ 0, & \text{otherwise.}\end{cases}$$

The dx^i's form a basis for the space of covectors: indeed, let φ be a linear map on the space of vectors, then

$$\varphi(\underbrace{X}_{X^i\partial_i}) = \varphi(X^i\partial_i) \underbrace{=}_{\text{linearity}} X^i \underbrace{\varphi(\partial_i)}_{\text{call this }\varphi_i} = \varphi_i dx^i(X) \underbrace{=}_{\text{def. of sum of functions}} (\varphi_i dx^i)(X)\,,$$

hence

$$\varphi = \varphi_i dx^i \,,$$

and every φ can indeed be written as a linear combination of the dx^i's. Under a change of coordinates we have

$$\bar\varphi_i \bar X^i = \bar\varphi_i \frac{\partial y^i}{\partial x^k} X^k = \varphi_k X^k \,,$$

leading to the following transformation law for components of covectors:

$$\varphi_k = \bar\varphi_i \frac{\partial y^i}{\partial x^k} \,. \tag{A.4.2}$$

Given a scalar f, we define its *differential* df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n \,.$$

With this definition, dx^i *is* the differential of the coordinate function x^i.

As presented above, the differential of a function is a covector by definition. As an exercice, you should check directly that the collection of functions $\varphi_i := \partial_i f$ satisfies the transformation rule (A.4.2).

We have a formula which is often used in calculations

$$dy^j = \frac{\partial y^j}{\partial x^k} dx^k \,.$$

INCIDENTALLY: An elegant approach to the definition of differentials proceeds as follows: Given any function f, we define:

$$df(X) := X(f) \,. \tag{A.4.3}$$

(Recall that here we are viewing a vector field X as a differential operator on functions, defined by (A.3.4).) The map $X \mapsto df(X)$ is linear under addition of vectors, and multiplication of vectors by numbers: if λ, μ are real numbers, and X and Y are vector fields, then

$$\begin{aligned} df(\lambda X + \mu Y) & \underbrace{=}_{\text{by definition (A.4.3)}} (\lambda X + \mu Y)(f) \\ & \underbrace{=}_{\text{by definition (A.3.4)}} \lambda X^i \partial_i f + \mu Y^i \partial_i f \\ & \underbrace{=}_{\text{by definition (A.4.3)}} \lambda df(X) + \mu df(Y) \,. \end{aligned}$$

Applying (A.4.3) to the function $f = x^i$ we obtain

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \,,$$

recovering (A.4.1). □

EXAMPLE A.4.2 Let (ρ, φ) be polar coordinates on $\mathbb{R}^2$, thus $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, and then

$$\begin{aligned} dx &= d(\rho\cos\varphi) = \cos\varphi d\rho - \rho\sin\varphi d\varphi\,, \\ dy &= d(\rho\sin\varphi) = \sin\varphi d\rho + \rho\cos\varphi d\varphi\,. \end{aligned}$$

At any given point $p \in M$, the set of covectors forms a vector space, denoted by T_p^*M. The collection of all the tangent spaces is called the cotangent bundle to M, denoted by T^*M.

Summarising, covectors are dual to vectors. It is convenient to define

$$\boxed{dx^i(X) := X^i}\,,$$

where X^i is as in (A.3.2). With this definition the (locally defined) bases $\{\partial_i\}_{i=1,\dots,\dim M}$ of TM and $\{dx^j\}_{i=1,\dots,\dim M}$ of T^*M are dual to each other:

$$\langle dx^i, \partial_j\rangle := dx^i(\partial_j) = \delta^i_j\,,$$

where δ^i_j is the Kronecker delta, equal to one when $i = j$ and zero otherwise.

A.5 Bilinear maps, two-covariant tensors

A map is said to be multi-linear if it is linear in every entry; e.g. g is bilinear if

$$g(aX + bY, Z) = ag(X, Z) + bg(Y, Z)\,,$$

and

$$g(X, aZ + bW) = ag(X, Z) + bg(X, W)\,.$$

Here, as elsewhere when talking about *tensors*, bilinearity is meant with respect to addition and to multiplication by functions.

A map g which is bilinear on the space of vectors can be represented by a matrix with two indices down:

$$g(X, Y) = g(X^i\partial_i, Y^j\partial_j) = X^iY^j \underbrace{g(\partial_i, \partial_j)}_{=:g_{ij}} = g_{ij}X^iY^j = g_{ij}dx^i(X)dx^j(Y)\,.$$

We say that g is a *covariant tensor of valence two.*

We say that g is *symmetric* if $g(X, Y) = g(Y, X)$ for all X, Y; equivalently, $g_{ij} = g_{ji}$.

A symmetric bilinear tensor field is said to be *non-degenerate* if $\det g_{ij}$ has no zeros.

By Sylvester's inertia theorem, there exists a basis θ^i of the space of covectors so that a symmetric bilinear map g can be written as

$$g(X, Y) = -\theta^1(X)\theta^1(Y) - \ldots - \theta^s(X)\theta^s(Y) + \theta^{s+1}(X)\theta^{s+1}(Y) + \ldots + \theta^{s+r}(X)\theta^{s+r}(Y)$$

(s, r) is called the signature of g; in geometry, unless specifically said otherwise, one always assumes that the signature does not change from point to point.

If $r = n$, in dimension n, then g is said to be a Riemannian metric tensor.

A canonical example is provided by the flat Riemannian metric on $\mathbb{R}^n$,

$$g = (dx^1)^2 + \ldots + (dx^n)^2 .$$

By definition, a *Riemannian metric* is a field of symmetric two-covariant tensors with signature $(+, \ldots, +)$ and with $\det g_{ij}$ without zeros.

INCIDENTALLY: A Riemannian metric can be used to define the length of curves: if $\gamma : [a, b] \ni s \to \gamma(s)$, then

$$\ell_g(\gamma) = \int_a^b \sqrt{g(\dot\gamma, \dot\gamma)} ds .$$

One can then define the distance between points by minimizing the length of the curves connecting them. □

If $s = 1$ and $r = N - 1$, in dimension N, then g is said to be a *Lorentzian metric tensor*.

For example, the *Minkowski metric* on $\mathbb{R}^{1+n}$ is

$$\eta = (dx^0)^2 - (dx^1)^2 - \ldots - (dx^n)^2 .$$

A.6 Tensor products

If φ and θ are covectors we can define a bilinear map using the formula

$$(\varphi \otimes \theta)(X, Y) = \varphi(X)\theta(Y) . \tag{A.6.1}$$

For example

$$(dx^1 \otimes dx^2)(X, Y) = X^1 Y^2 .$$

Using this notation we have

$$g(X, Y) = g(X^i \partial_i, Y^j \partial_j) = \underbrace{g(\partial_j, \partial_j)}_{=:g_{ij}} \underbrace{\underbrace{X^i}_{dx^i(X)} \underbrace{Y^j}_{dx^j(Y)}}_{(dx^i \otimes dx^j(X,Y)} = (g_{ij} dx^i \otimes dx^j)(X, Y)$$

We will write $dx^i dx^j$ for the symmetric product,

$$dx^i dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i) ,$$

and $dx^i \wedge dx^j$ for twice the anti-symmetric one (compare Section A.15):

$$dx^i \wedge dx^j := dx^i \otimes dx^j - dx^j \otimes dx^i .$$

It should be clear how this generalises: the tensors $dx^i \otimes dx^j \otimes dx^k$, defined as

$$(dx^i \otimes dx^j \otimes dx^k)(X, Y, Z) = X^i Y^j Z^k ,$$

form a basis of three-linear maps on the space of vectors, which are objects of the form

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k \,.$$

Here X is a called *tensor of valence* $(0, 3)$. Each index transforms as for a covector:

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k = X_{ijk} \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^\ell} \frac{\partial x^k}{\partial y^n} dy^m \otimes dy^\ell \otimes dy^n \,.$$

It is sometimes useful to think of vectors as linear maps on co-vectors, using a formula which looks funny when first met: if θ is a covector, and X is a vector, then

$$X(\theta) := \theta(X) \,.$$

So if $\theta = \theta_i dx^i$ and $X = X^i \partial_i$ then

$$\theta(X) = \theta_i X^i = X^i \theta_i = X(\theta) \,.$$

It then makes sense to define e.g. $\partial_i \otimes \partial_j$ as a bilinear map on covectors:

$$(\partial_i \otimes \partial_j)(\theta, \psi) := \theta_i \psi_j \,.$$

And one can define a map $\partial_i \otimes dx^j$ which is linear on forms in the first slot, and linear in vectors in the second slot as

$$(\partial_i \otimes dx^j)(\theta, X) := \partial_i(\theta) dx^j(X) = \theta_i X^j \,. \tag{A.6.2}$$

The $\partial_i \otimes dx^j$'s form the basis of the space of *tensors of rank* $(1, 1)$:

$$T = T^i{}_j \partial_i \otimes dx^j \,.$$

Generally, a *tensor of valence, or rank,* (r, s) can be defined as an object which has r vector indices and s covector indices, so that it transforms as

$$S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \to S^{m_1 \ldots m_r}{}_{\ell_1 \ldots \ell_s} \frac{\partial y^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial y^{i_r}}{\partial x^{m_r}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_s}}{\partial y^{j_s}}$$

For example, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$ are vectors, then $X \otimes Y = X^i Y^j \partial_i \otimes \partial_j$ forms a contravariant tensor of valence two.

Tensors of same valence can be added in the obvious way: *e.g.*

$$(A + B)(X, Y) := A(X, Y) + B(X, Y) \quad \Longleftrightarrow \quad (A + B)_{ij} = A_{ij} + B_{ij} \,.$$

Tensors can be multiplied by scalars: *e.g.*

$$(fA)(X, Y, Z) := fA(X, Y, Z) \quad \Longleftrightarrow \quad f(A_{ijk}) := (fA_{ijk}) \,.$$

Finally, we have seen in (A.6.1) how to take tensor products for one-forms, and in (A.6.2) how to take a tensor product of a vector and a one-form, but this can also be done for higher order tensor; e.g., if S is of valence (a, b) and T is a multilinear map of valence (c, d), then $S \otimes T$ is a multilinear map of valence $(a + c, b + d)$, defined as

$$(S \otimes T)(\underbrace{\theta, \ldots}_{a \text{ covectors and } b \text{ vectors}}, \underbrace{\psi, \ldots}_{c \text{ covectors and } d \text{ vectors}}) := S(\theta, \ldots) T(\psi, \ldots) \,.$$

A.6.1 Contractions

Given a tensor field $S^i{}_j$ with one index down and one index up one can perform the sum

$$S^i{}_i\,.$$

This defines a scalar, i.e., a function on the manifold. Indeed, using the transformation rule

$$S^i{}_j \to \bar S^\ell{}_k = S^i{}_j \frac{\partial x^j}{\partial y^k}\frac{\partial y^\ell}{\partial x^i}\,,$$

one finds

$$\bar S^\ell{}_\ell = S^i{}_j \underbrace{\frac{\partial x^j}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^i}}_{\delta^j_i} = S^i{}_i\,,$$

as desired.

One can similarly do contractions on higher valence tensors, e.g.

$$S^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 j_3 \ldots j_s} \to S^{\ell i_2 \ldots i_r}{}_{j_1 \ell j_3 \ldots j_s}\,.$$

After contraction, a tensor of rank $(r+1, s+1)$ becomes of rank (r, s).

A.7 Raising and lowering of indices

Let g be a symmetric two-covariant tensor field on M, by definition such an object is the assignment to each point $p \in M$ of a bilinear map $g(p)$ from $T_pM \times T_pM$ to $\mathbb{R}$, with the additional property

$$g(X, Y) = g(Y, X)\,.$$

In this work the symbol g will be reserved to *non-degenerate* symmetric two-covariant tensor fields. It is usual to simply write g for $g(p)$, the point p being implicitly understood. We will sometimes write g_p for $g(p)$ when referencing p will be useful.

The usual Sylvester's inertia theorem tells us that at each p the map g will have a well defined signature; clearly this signature will be point-independent on a connected manifold when g is non-degenerate. A pair (M, g) is said to be a *Riemannian manifold* when the signature of g is $(\dim M, 0)$; equivalently, when g is a positive definite bilinear form on every product $T_pM \times T_pM$. A pair (M, g) is said to be a *Lorentzian manifold* when the signature of g is $(\dim M - 1, 1)$. One talks about *pseudo-Riemannian* manifolds whatever the signature of g, as long as g is non-degenerate, but we will only encounter Riemannian and Lorentzian metrics in this work.

Since g is non-degenerate it induces an isomorphism

$$\flat : T_pM \to T_p^*M$$

by the formula

$$\boxed{X_\flat(Y) = g(X, Y)}\,.$$

In local coordinates this gives

$$X_\flat = g_{ij} X^i dx^j =: X_j dx^j \,. \tag{A.7.1}$$

This last equality defines X_j — "the vector X^j with the index j lowered":

$$\boxed{X_i := g_{ij} X^j}\,. \tag{A.7.2}$$

The operation (A.7.2) is called the *lowering of indices* in the physics literature and, again in the physics literature, one does not make a distinction between the one-form $X_\flat$ and the vector X.

The inverse map will be denoted by $\sharp$ and is called the *raising of indices*; from (A.7.1) we obviously have

$$\alpha^\sharp = g^{ij}\alpha_i\partial_j =: \alpha^i\partial_i \qquad \Longleftrightarrow \qquad dx^i(\alpha^\sharp) = \boxed{\alpha^i = g^{ij}\alpha_j}\,,$$

where g^{ij} is the matrix inverse to g_{ij}. For example,

$$(dx^i)^\sharp = g^{ik}\partial_k \,.$$

Clearly g^{ij}, understood as the matrix of a bilinear form on T^*_pM, has the same signature as g, and can be used to define a scalar product $g^\sharp$ on $T^*_p(M)$:

$$g^\sharp(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp) \qquad \Longleftrightarrow \qquad g^\sharp(dx^i,dx^j) = g^{ij}\,.$$

This last equality is justified as follows:

$$g^\sharp(dx^i,dx^j) = g((dx^i)^\sharp,(dx^j)^\sharp) = g(g^{ik}\partial_k, g^{j\ell}\partial_\ell) = \underbrace{g^{ik}g_{k\ell}}_{=\delta^i_\ell} g^{j\ell} = g^{ji} = g^{ij}\,.$$

It is convenient to use the same letter g for $g^\sharp$ — physicists do it all the time — or for scalar products induced by g on all the remaining tensor bundles, and we will sometimes do so.

INCIDENTALLY: One might wish to check by direct calculations that $g_{\mu\nu}X^\nu$ transforms as a one-form if X^μ transforms as a vector. The simplest way is to notice that $g_{\mu\nu}X^\nu$ is a contraction, over the last two indices, of the three-index tensor $g_{\mu\nu}X^\alpha$. Hence it is a one-form by the analysis at the end of the previous section. Alternatively, if we write $\bar g_{\mu\nu}$ for the transformed $g_{\mu\nu}$'s, and $\bar X^\alpha$ for the transformed X^α's, then

$$\underbrace{\bar g_{\alpha\beta}}_{g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}} \bar X^\beta = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\underbrace{\frac{\partial x^\nu}{\partial y^\beta}\bar X^\beta}_{X^\nu} = g_{\mu\nu}X^\nu\frac{\partial x^\mu}{\partial y^\alpha}\,,$$

which is indeed the transformation law of a covector. □

The *gradient* ∇f of a function f is a vector field obtained by raising the indices on the differential df:

$$g(\nabla f, Y) := df(Y) \qquad \Longleftrightarrow \qquad \nabla f := g^{ij}\partial_i f \partial_j \,. \tag{A.7.3}$$

A.8 The Lie derivative

A.8.1 A pedestrian approach

We start with a pedestrian approach to the definition of Lie derivative; the elegant geometric definition will be given in the next section.

Given a vector field X, the *Lie derivative* $\mathscr{L}_X$ is an operation on tensor fields, defined as follows:

For a function f, one sets

$$\mathscr{L}_X f := X(f)\,. \tag{A.8.1}$$

For a vector field Y, the Lie derivative coincides with the Lie bracket:

$$\mathscr{L}_X Y := [X,Y]\,. \tag{A.8.2}$$

For a one-form α, $\mathscr{L}_X\alpha$ is defined by imposing the Leibniz rule written the wrong-way round:

$$(\mathscr{L}_X\alpha)(Y) := \mathscr{L}_X(\alpha(Y)) - \alpha(\mathscr{L}_X Y)\,. \tag{A.8.3}$$

(Indeed, the Leibniz rule applied to the contraction $\alpha_i X^i$ would read

$$\mathscr{L}_X(\alpha_i Y^i) = (\mathscr{L}_X\alpha)_i Y^i + \alpha_i(\mathscr{L}_X Y)^i\,,$$

which can be rewritten as (A.8.3).)

Let us check that (A.8.3) defines a one-form. Clearly, the right-hand side transforms in the desired way when Y is replaced by Y_1+Y_2. Now, if we replace Y by fY, where f is a function, then

$$\begin{aligned}
(\mathscr{L}_X\alpha)(fY) &= \mathscr{L}_X(\alpha(fY)) - \alpha(\underbrace{\mathscr{L}_X(fY)}_{X(f)Y+f\mathscr{L}_X Y})\\
&= X(f\alpha(Y)) - \alpha(X(f)Y + f\mathscr{L}_X Y))\\
&= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y) - \alpha(f\mathscr{L}_X Y))\\
&= fX(\alpha(Y)) - f\alpha(\mathscr{L}_X Y))\\
&= f((\mathscr{L}_X\alpha)(Y))\,.
\end{aligned}$$

So $\mathscr{L}_X\alpha$ is a C^∞-linear map on vector fields, hence a covector field.

In coordinate-components notation we have

$$(\mathscr{L}_X\alpha)_a = X^b\partial_b\alpha_a + \alpha_b\partial_a X^b\,. \tag{A.8.4}$$

Indeed,

$$\begin{aligned}
(\mathscr{L}_X\alpha)_i Y^i &:= \mathscr{L}_X(\alpha_i Y^i) - \alpha_i(\mathscr{L}_X Y)^i\\
&= X^k\partial_k(\alpha_i Y^i) - \alpha_i(X^k\partial_k Y^i - Y^k\partial_k X^i)\\
&= X^k(\partial_k\alpha_i)Y^i + \alpha_i Y^k\partial_k X^i\\
&= \left(X^k\partial_k\alpha_i + \alpha_k\partial_i X^k\right) Y^i\,,
\end{aligned}$$

as desired

For tensor products, the Lie derivative is defined by imposing linearity under addition together with the Leibniz rule:

$$\mathscr{L}_X(\alpha \otimes \beta) = (\mathscr{L}_X \alpha) \otimes \beta + \alpha \otimes \mathscr{L}_X \beta \,.$$

Since a general tensor A is a sum of tensor products,

$$A = A^{a_1 \dots a_p}{}_{b_1 \dots b_q} \partial_{a_1} \otimes \dots \partial_{a_p} \otimes dx^{b_1} \otimes \dots \otimes dx^{a_p} \,,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

For example, we claim that

$$\mathscr{L}_X T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c \,, \tag{A.8.5}$$

To see this, call a tensor $T^a{}_b$ *simple* if it is of the form $Y \otimes \alpha$, where Y is a vector and α is a covector. Using indices, this corresponds to $Y^a \alpha_b$ and so, by the Leibniz rule,

$$\begin{aligned}
\mathscr{L}_X (Y \otimes \alpha)^a{}_b &= \mathscr{L}_X (Y^a \alpha_b) \\
&= (\mathscr{L}_X Y)^a \alpha_b + Y^a (\mathscr{L}_X \alpha)_b \\
&= (X^c \partial_c Y^a - Y^c \partial_c X^a) \alpha_b + Y^a (X^c \partial_c \alpha_b + \alpha_c \partial_b X^c) \\
&= X^c \partial_c (Y^a \alpha_b) - Y^c \alpha_b \partial_c X^a + Y^a \alpha_c \partial_b X^c \,,
\end{aligned}$$

which coincides with (A.8.5) if $T^a{}_b = Y^b \alpha_b$. But a general $T^a{}_b$ can be written as a linear combination with constant coefficients of simple tensors,

$$T = \sum_{a,b} \underbrace{T^a{}_b \partial_a \otimes dx^b}_{\text{no summation, so simple}} \,,$$

and the result follows.

Similarly, one has, e.g.,

$$\begin{aligned}
\mathscr{L}_X R^{ab} &= X^c \partial_c R^{ab} - R^{ac} \partial_c X^b - R^{bc} \partial_c X^a \,, &\text{(A.8.6)} \\
\mathscr{L}_X S_{ab} &= X^c \partial_c S_{ab} + S_{ac} \partial_b X^c + S_{bc} \partial_a X^c \,. &\text{(A.8.7)}
\end{aligned}$$

etc. Those are all special cases of the general formula for the Lie derivative $\mathscr{L}_X A^{a_1 \dots a_p}{}_{b_1 \dots b_q}$:

$$\begin{aligned}
\mathscr{L}_X A^{a_1 \dots a_p}{}_{b_1 \dots b_q} &= X^c \partial_c A^{a_1 \dots a_p}{}_{b_1 \dots b_q} - A^{c a_2 \dots a_p}{}_{b_1 \dots b_q} \partial_c X^{a_1} - \dots \\
&\quad + A^{a_1 \dots a_p}{}_{c b_1 \dots b_q} \partial_{b_1} X^c + \dots \,.
\end{aligned}$$

A useful property of Lie derivatives is

$$\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y] \,, \tag{A.8.8}$$

where, for a tensor T, the commutator $[\mathscr{L}_X, \mathscr{L}_Y]T$ is defined in the usual way:

$$[\mathscr{L}_X, \mathscr{L}_Y]T := \mathscr{L}_X(\mathscr{L}_Y T) - \mathscr{L}_Y(\mathscr{L}_X T) \,. \tag{A.8.9}$$

To see this, we first note that if $T = f$ is a function, then the right-hand side of (A.8.9) is the definition of $[X,Y](f)$, which in turn coincides with the definition of $\mathscr{L}_{[X,Y]}(f)$.

Next, for a vector field $T = Z$, (A.8.8) reads

$$\mathscr{L}_{[X,Y]}Z = \mathscr{L}_X(\mathscr{L}_Y Z) - \mathscr{L}_Y(\mathscr{L}_X Z)\,, \tag{A.8.10}$$

which is the same as

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]\,, \tag{A.8.11}$$

which is the same as

$$[Z,[Y,X]] + [X,[Z,Y]] + [Y,[X,Z]] = 0\,, \tag{A.8.12}$$

which is the Jacobi identity. Hence (A.8.8) holds for vector fields.

We continue with a one-form α, exploiting the fact that we have already established the result for functions and vectors: For any vector field Z we have, by definition

$$\begin{aligned}
([\mathscr{L}_X,\mathscr{L}_Y]\alpha)(Z) &= [\mathscr{L}_X,\mathscr{L}_Y](\alpha(Z)) - \alpha([\mathscr{L}_X,\mathscr{L}_Y](Z)) \\
&= \mathscr{L}_{[X,Y]}(\alpha(Z)) - \alpha(\mathscr{L}_{[X,Y]}(Z)) \\
&= (\mathscr{L}_{[X,Y]}\alpha)(Z)\,.
\end{aligned}$$

INCIDENTALLY: A direct calculation for one-forms, using the definitions, proceed as follows: Let Z be any vector field,

$$\begin{aligned}
(\mathscr{L}_X\mathscr{L}_Y\alpha)(Z) &= X\Big(\underbrace{(\mathscr{L}_Y\alpha)(Z)}_{Y(\alpha(Z))-\alpha(\mathscr{L}_Y Z)}\Big) - \underbrace{(\mathscr{L}_Y\alpha)\,(\mathscr{L}_X Z)}_{Y(\alpha(\mathscr{L}_X Z))-\alpha(\mathscr{L}_Y\mathscr{L}_X Z)} \\
&= X\big(Y(\alpha(Z))\big) - X\big(\alpha(\mathscr{L}_Y Z)\big) - Y\left(\alpha(\mathscr{L}_X Z)\right) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z)\,.
\end{aligned}$$

Antisymmetrizing over X and Y, the second and third term above cancel out, so that

$$\begin{aligned}
\big((\mathscr{L}_X\mathscr{L}_Y\alpha - \mathscr{L}_Y\mathscr{L}_X)\alpha\big)(Z) &= X\big(Y(\alpha(Z))\big) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z) - (X \longleftrightarrow Y) \\
&= [X,Y]\big(\alpha(Z)\big) - \alpha(\mathscr{L}_X\mathscr{L}_Y Z - \mathscr{L}_Y\mathscr{L}_X Z) \\
&= \mathscr{L}_{[X,Y]}\big(\alpha(Z)\big) - \alpha(\mathscr{L}_{[X,Y]}Z) \\
&= \big(\mathscr{L}_{[X,Y]}\alpha\big)(Z)\,.
\end{aligned}$$

Since Z is arbitrary, (A.8.8) for covectors follows. □

To conclude that (A.8.8) holds for arbitrary tensor fields, we note that by construction we have

$$\mathscr{L}_{[X,Y]}(A\otimes B) = \mathscr{L}_{[X,Y]}A\otimes B + A\otimes\mathscr{L}_{[X,Y]}B\,. \tag{A.8.13}$$

Similarly

$$\begin{aligned}
\mathscr{L}_X\mathscr{L}_Y(A\otimes B) &= \mathscr{L}_X(\mathscr{L}_Y A\otimes B + A\otimes\mathscr{L}_Y B) \\
&= \mathscr{L}_X\mathscr{L}_Y A\otimes B + \mathscr{L}_X A\otimes\mathscr{L}_Y B + \mathscr{L}_Y A\otimes\mathscr{L}_X B \\
&\quad + A\otimes\mathscr{L}_X\mathscr{L}_Y B\,.
\end{aligned} \tag{A.8.14}$$

Exchanging X with Y and subtracting, the middle terms drop out:

$$[\mathscr{L}_X, \mathscr{L}_Y](A \otimes B) = [\mathscr{L}_X, \mathscr{L}_Y]A \otimes B + A \otimes [\mathscr{L}_X, \mathscr{L}_Y]B\,. \tag{A.8.15}$$

Basing on what has been said, the reader should have no difficulties finishing the proof of (A.8.8).

EXAMPLE A.8.2 As an example of application of the formalism, suppose that there exists a coordinate system in which $(X^a) = (1, 0, 0, 0)$ and $\partial_0 g_{bc} = 0$. Then

$$\mathscr{L}_X g_{ab} = \partial_0 g_{ab} = 0\,.$$

But the Lie derivative of a tensor field is a tensor field, and we conclude that $\mathscr{L}_X g_{ab} = 0$ holds in every coordinate system.

Vector fields for which $\mathscr{L}_X g_{ab} = 0$ are called *Killing vectors*: they arise from symmetries of spacetime. We have the useful formula

$$\boxed{\mathscr{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a\,.} \tag{A.8.16}$$

An effortless proof of this proceeds as follows: in adapted coordinates in which the derivatives of the metric vanish at a point p, one immediately checks that equality holds at p. But both sides are tensor fields, therefore the result holds at p for all coordinate systems, and hence also everywhere.

The brute-force proof of (A.8.16) proceeds as follows:

$$\begin{aligned}
\mathscr{L}_X g_{ab} &= X^c \partial_c g_{ab} + \partial_a X^c g_{cb} + \partial_b X^c g_{ca} \\
&= X^c \partial_c g_{ab} + \partial_a (X^c g_{cb}) - X^c \partial_a g_{cb} + \partial_b (X^c g_{ca}) - X^c \partial_b g_{ca} \\
&= \partial_a X_b + \partial_b X_a + X^c \underbrace{(\partial_c g_{ab} - \partial_a g_{cb} - \partial_b g_{ca})}_{-2 g_{cd} \Gamma^d_{ab}} \\
&= \nabla_a X_b + \nabla_b X_a\,.
\end{aligned}$$

□

A.8.2 The geometric approach

We pass now to a geometric definition of Lie derivative. This requires, first, an excursion through the land of push-forwards and pull-backs.

Transporting tensor fields

We start by noting that, given a point p_0 in a manifold M, every vector $X \in T_{p_0}M$ is tangent to some curve. To see this, let $\{x^i\}$ be any local coordinates near p_0, with $x^i(p_0) = x^i_0$, then X can be written as $X^i(p_0)\partial_i$. If we set $\gamma^i(s) = x^i_0 + sX^i(p_0)$, then $\dot\gamma^i(0) = X^i(p_0)$, which establishes the claim. This observation shows that studies of vectors can be reduced to studies of curves.

Let, now, M and N be two manifolds, and let $\phi : M \to N$ be a differentiable map between them. Given a vector $X \in T_pM$, the *push-forward* $\phi_* X$ of X is a vector in $T_{\phi(p)}N$ defined as follows: let γ be any curve for which $X = \dot\gamma(0)$, then

$$\phi_* X := \left.\frac{d(\phi \circ \gamma)}{ds}\right|_{s=0}. \tag{A.8.17}$$

In local coordinates y^A on N and x^i on M, so that $\phi(x) = (\phi^A(x^i))$, we find

$$\begin{aligned}(\phi_* X)^A &= \left.\frac{d\phi^A(\gamma^i(s))}{ds}\right|_{s=0} = \frac{\partial\phi^A(\gamma^i(s))}{\partial x^i}\dot\gamma^i(s)|_{s=0} \\ &= \frac{\partial\phi^A(x^i)}{\partial x^i}X^i \,. \end{aligned} \tag{A.8.18}$$

The formula makes it clear that the definition is independent of the choice of the curve γ satisfying $X = \dot\gamma(0)$.

Equivalently, and more directly, if X is a vector at p and h is a function on h, then $\phi^* X$ acts on h as

$$\boxed{\phi^* X(h) := X(h\circ\phi)\,.} \tag{A.8.19}$$

Applying (A.8.18) to a *vector field* X defined on M we obtain

$$(\phi_* X)^A(\phi(x)) = \frac{\partial\phi^A}{\partial x^i}(x)\, X^i(x)\,. \tag{A.8.20}$$

The equation shows that if a point $y\in N$ has more than one pre-image, say $y = \phi(x_1) = \phi(x_2)$ with $x_1 \neq x_2$, then (A.8.20) might will define more than one tangent vector at y in general. This leads to an important caveat: we will be certain that the push-forward *of a vector field* on M defines a vector field on N only when ϕ is a diffeomorphism. More generally, $\phi_* X$ defines locally a vector field on $\phi(M)$ if and only if ϕ is a local diffeomorphism. In such cases we can invert ϕ (perhaps locally) and write (A.8.20) as

$$(\phi_* X)^j(x) = \left(\frac{\partial\phi^j}{\partial x^i}X^i\right)(\phi^{-1}(x))\,. \tag{A.8.21}$$

When ϕ is understood as a coordinate change rather than a diffeomorphism between two manifolds, this is simply the standard transformation law of a vector field under coordinate transformations.

The push-forward operation can be extended to *contravariant* tensors by defining it on tensor products in the obvious way, and extending by linearity: for example, if X, Y and Z are vectors, then

$$\phi_*(X\otimes Y\otimes Z) := \phi_* X\otimes\phi_* Y\otimes\phi_* Z\,.$$

Consider, next, a k-multilinear map α from $T_{\phi(p_0)}M$ to $\mathbb{R}$. The *pull-back* $\phi^*\alpha$ of α is a multilinear map on $T_{p_0}M$ defined as

$$T_pM \ni (X_1,\ldots X_k)\mapsto \phi^*(\alpha)(X_1,\ldots,X_k) := \alpha(\phi^* X_1,\ldots,\phi_* X_k)\,.$$

As an example, let $\alpha = \alpha_A dy^A$ be a one-form. If $X = X^i\partial_i$ then

$$\begin{aligned}&\boxed{(\phi^*\alpha)(X) = \alpha(\phi_* X)} \\ &= \alpha(\frac{\partial\phi^A}{\partial x^i}X^i\partial_A) = \alpha_A\frac{\partial\phi^A}{\partial x^i}X^i = \alpha_A\frac{\partial\phi^A}{\partial x^i}dx^i(X)\,.\end{aligned} \tag{A.8.22}$$

Equivalently,

$$(\phi^*\alpha)_i = \alpha_A \frac{\partial \phi^A}{\partial x^i}\,. \tag{A.8.23}$$

If α is a *one-form field* on N, this reads

$$(\phi^*\alpha)_i(x) = \alpha_A(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i}\,. \tag{A.8.24}$$

It follows that $\phi^*\alpha$ is a field of one-forms on M, *irrespective of injectivity or surjectivity* properties of ϕ. Similarly, pull-backs of covariant tensor fields of higher rank are smooth tensor fields.

For a function f equation (A.8.24) reads

$$(\phi^* df)_i(x) = \frac{\partial f}{\partial y^A}(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i} = \frac{\partial (f\circ\phi)}{\partial x^i}(x)\,, \tag{A.8.25}$$

which can be succinctly written as

$$\phi^* df = d(f\circ \phi)\,. \tag{A.8.26}$$

Using the notation

$$\phi^* f := f\circ\phi\,, \tag{A.8.27}$$

we can write (A.8.26) as

$$\phi^* d = d\phi^* \quad \text{for functions.} \tag{A.8.28}$$

Summarising:

1. Pull-backs of covariant tensor fields define covariant tensor fields. In particular the metric can *always* be pulled back.

2. Push-forwards of contravariant tensor fields can be used to define contravariant tensor fields when ϕ is a diffeomorphism.

In this context it is thus clearly of interest to consider diffeomorphisms ϕ, as then tensor products can now be transported in the following way; we will denote by $\hat\phi$ the associated map: We define $\hat\phi f := f\circ\phi$ for functions, $\hat\phi := \phi_*$ for covariant fields, $\hat\phi := (\phi^{-1})_*$ for contravariant tensor fields. We use the rule

$$\hat\phi(A\otimes B) = \hat\phi A \otimes \hat\phi B$$

for tensor products.

So, for example, if X is a vector field and α is a field of one-forms, one has

$$\hat\phi(X\otimes\alpha) := (\phi^{-1})_* X \otimes \phi^*\alpha\,. \tag{A.8.29}$$

The definition is extended by linearity under addition and multiplication by functions to any tensor fields. Thus, if f is a function and T and S are tensor fields, then

$$\hat\phi(f\,T + S) = \hat\phi f\,\hat\phi T + \hat\phi S \equiv f\circ\phi\,\hat\phi T + \hat\phi S\,.$$

Since everything was fairly natural so far, one would expect that contractions transform in a natural way under transport. To make this clear, we start by rewriting (A.8.22) with the base-points made explicit:

$$((\hat{\phi}\alpha)(X))(x) = (\alpha(\phi^* X))(\phi(x))\,. \tag{A.8.30}$$

Replacing X by $(\phi^*)^{-1}Y$ this becomes

$$((\hat{\phi}\alpha)(\hat{\phi}Y))(x) = (\alpha(Y))(\phi(x))\,. \tag{A.8.31}$$

Equivalently

$$(\hat{\phi}\alpha)(\hat{\phi}Y) = \hat{\phi}(\alpha(Y))\,. \tag{A.8.32}$$

Flows of vector fields

Let X be a vector field on M. For every $p_0 \in M$ consider the solution to the problem

$$\frac{dx^i}{dt} = X^i(x(t))\,, \quad x^i(0) = x^i_0\,. \tag{A.8.33}$$

(Recall that there always exists a *maximal* interval I containing the origin on which (A.8.33) has a solution. Both the interval and the solution are unique. This will always be the solution $I \ni t \mapsto x(t)$ that we will have in mind.) The map

$$(t, x_0) \mapsto \phi_t[X](x_0) := x(t)$$

where $x^i(t)$ is the solution of (A.8.33), is called *the local flow of X*. We say that X generates $\phi_t[X]$. We will write ϕ_t for $\phi_t[X]$ when X is unambiguous in the context.

The interval of existence of solutions of (A.8.33) depends upon x_0 in general.

Example A.8.3 As an example, let $M = \mathbb{R}$ and $X = x^2\partial_x$. We then have to solve

$$\frac{dx}{dt} = x^2\,,\ x(0) = x_0 \quad \Longrightarrow \quad x(t) = \begin{cases} 0, & x_0 = 0; \\ \frac{x_0}{1-x_0 t}, & x_0 \neq 0\,,\ 1 - x_0 t > 0. \end{cases}$$

Hence

$$\phi_t(x) = \frac{x}{1-xt}\,,$$

with $t \in \mathbb{R}$ when $x = 0$, with $t \in (-\infty, 1/x)$ when $x > 0$ and with $t \in (1/x, \infty)$ when $x < 0$. □

We say that X is *complete* if $\phi_t[X](p)$ is defined for all $(t, p) \in \mathbb{R} \times M$.

The following standard facts are left as exercices to the reader:

1. ϕ_0 is the identity map.
2. $\phi_t \circ \phi_s = \phi_{t+s}$.
   In particular, $\phi_t^{-1} = \phi_{-t}$, and thus:
3. The maps $x \mapsto \phi_t(x)$ are local diffeomorphisms; global if for all $x \in M$ the maps ϕ_t are defined for all $t \in \mathbb{R}$.

4. $\phi_{-t}[X]$ is generated by $-X$:

$$\phi_{-t}[X] = \phi_t[-X] \,.$$

A family of diffeomorphisms satisfying property 2. above is called a *one parameter group of diffeomorphisms.* Thus, *complete* vector fields generate one-parameter families of diffeomorphisms via (A.8.33).

Reciprocally, suppose that a local or global one-parameter group ϕ_t is given, then the formula

$$X = \frac{d\phi_t}{dt}\bigg|_{t=0}$$

defines a vector field, said to be *generated by* ϕ_t.

The Lie derivative revisited

The idea of the *Lie transport*, and hence of the *Lie derivative*, is to be able to compare objects along integral curves of a vector field X. This is pretty obvious for scalars: we just compare the values of $f(\phi_t(x))$ with $f(x)$, leading to a derivative

$$\mathscr{L}_X f := \lim_{t\to 0} \frac{f\circ\phi_t - f}{t} \equiv \lim_{t\to 0} \frac{\phi_t^* f - f}{t} \equiv \lim_{t\to 0} \frac{\hat\phi_t f - f}{t} \equiv \frac{d(\hat\phi_t f)}{dt}\bigg|_{t=0} \,. \quad \text{(A.8.34)}$$

We wish, next, to compare the value of a vector field Y at $\phi_t(x)$ with the value at x. For this, we move from x to $\phi_t(x)$ following the integral curve of X, and produce a new vector at x by applying $(\phi_t^{-1})_*$ to $Y|_{\phi_t(x)}$. This makes it perhaps clearer why we introduced the transport map $\hat\phi$, since $(\hat\phi Y)(x)$ is precisely the value at x of $(\phi_t^{-1})_* Y$. We can then calculate

$$\mathscr{L}_X Y(x) := \lim_{t\to 0} \frac{((\phi_t^{-1})*Y)(\phi_t(x)) - Y(x)}{t} \equiv \lim_{t\to 0} \frac{(\hat\phi_t Y)(x) - Y(x)}{t} \equiv \frac{d(\hat\phi_t Y(x))}{dt}\bigg|_{t=0} \,. \quad \text{(A.8.35)}$$

In general, let X be a vector field and let ϕ_t be the associated local one-parameter family of diffeomorphisms. Let $\hat\phi_t$ be the associated family of transport maps for tensor fields. For any tensor field T one sets

$$\mathscr{L}_X T := \lim_{t\to 0} \frac{\hat\phi_t T - T}{t} \equiv \frac{d(\hat\phi_t T)}{dt}\bigg|_{t=0} \,. \quad \text{(A.8.36)}$$

We want to show that this operation coincides with that defined in Section A.8.1.

The equality of the two operations for functions should be clear, since (A.8.34) easily implies:

$$\mathscr{L}_X f = X(f) \,.$$

Consider, next, a vector field Y. From (A.8.21), setting $\psi_t := \phi_{-t} \equiv (\phi_t)^{-1}$ we have

$$\hat\phi_t Y^j(x) := ((\phi_t^{-1})_* Y)^j(x) = (\frac{\partial \psi_t^j}{\partial x^i} Y^i)(\phi_t(x)) \,. \quad \text{(A.8.37)}$$

Since ϕ_{-t} is generated by $-X$, we have

$$\psi^i_0(x) = x^i\,, \quad \frac{\partial \psi^j_t}{\partial x^i}\Big|_{t=0} = \delta^j_i\,,$$
$$\dot\psi^j_t|_{t=0} := \frac{d\psi^j_t}{dt}\Big|_{t=0} = -X^j\,, \quad \frac{\partial \dot\psi^j_t}{\partial x^i}\Big|_{t=0} = -\partial_i X^j\,. \tag{A.8.38}$$

Hence

$$\begin{aligned}\frac{d(\hat\phi_t Y^j)}{dt}(x)|_{t=0} &= \frac{\partial \dot\psi^j_0}{\partial x^i}(x)\,Y^i(x) + \partial_k(\underbrace{\frac{\partial \psi^j_0}{\partial x^i}\,Y^i}_{Y^j})(x)\dot\phi^k(x)\\ &= -\partial_i X^j(x)Y^i(x) + \partial_j Y^i(x)X^j(x)\\ &= [X,Y]^j(x)\,,\end{aligned}$$

and we have obtained (A.8.2), p. 238.

For a covector field α, it seems simplest to calculate directly from (A.8.24):

$$(\hat\phi_t\alpha)_i(x) = (\phi^*_t\alpha)_i(x) = \alpha_k(\phi_t(x))\frac{\partial\phi^k_t(x)}{\partial x^i}\,.$$

Hence

$$\mathscr{L}_X\alpha_i = \frac{d(\phi^*_t\alpha)_i(x)}{dt}\Big|_{t=0} = \partial_j\alpha_i(x)X^j(x) + \alpha_k(x)\frac{\partial X^k(x)}{\partial x^i}(x)\,, \tag{A.8.39}$$

as in (A.8.4).

The formulae just derived show that the *Leibniz rule under duality* holds by inspection:

$$\mathscr{L}_X(\alpha(Y)) = \mathscr{L}_X\alpha(Y) + \alpha(\mathscr{L}_X(Y))\,. \tag{A.8.40}$$

Incidentally: Alternatively, one can start by showing that the Leibniz rule under duality holds for (A.8.36), and then use the calculations in Section A.8.1 to derive (A.8.39): Indeed, by definition we have

$$\phi^*_t\alpha(Y) = \alpha((\phi_t)_*Y)\,,$$

hence

$$\alpha(Y)|_{\phi_t(x)} = \alpha((\phi_t)_*(\phi^{-1}_t)_*Y)|_{\phi_t(x)} = \phi^*_t\alpha|_x((\phi^{-1}_t)_*Y|_{\phi_t(x)}) = \hat\phi_t\alpha(\hat\phi_t Y)|_x\,.$$

Equivalently,

$$\hat\phi_t(\alpha(Y)) = (\hat\phi_t\alpha)(\hat\phi_t Y)\,,$$

from which the Leibniz rule under duality immediately follows.

A similar calculation leads to the Leibniz rule under tensor products. □

The reader should have no difficulties checking that the remaining requirements set forth in Section A.8.1 are satisfied.

The following formula of Cartan provides a convenient tool for calculating the Lie derivative of a differential form α:

$$\mathscr{L}_X\alpha = X\rfloor d\alpha + d(X\rfloor\alpha)\,. \tag{A.8.41}$$

The commuting of d and $\mathscr{L}_X$ is an immediate consequence of (A.8.41) and of the identity $d^2 = 0$:

$$\mathscr{L}_X d\alpha = d(\mathscr{L}_X\alpha)\,. \tag{A.8.42}$$

A.9 Covariant derivatives

When dealing with $\mathbb{R}^n$, or subsets thereof, there exists an obvious prescription for how to differentiate tensor fields: in this case we have at our disposal the canonical "trivialization $\{\partial_i\}_{i=1,\ldots,n}$ of $T\mathbb{R}^n$" (this means: a globally defined set of vectors which, at every point, form a basis of the tangent space), together with its dual trivialization $\{dx^j\}_{j=1,\ldots,n}$ of $T^*\mathbb{R}^n$. We can expand a tensor field T of valence (k,ℓ) in terms of those bases,

$$\begin{aligned} T &= T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \\ &\Longleftrightarrow \quad T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell} = T(dx^{i_1},\ldots,dx^{i_k},\partial_{j_1},\ldots,\partial_{j_\ell})\,, \end{aligned} \tag{A.9.1}$$

and differentiate each component $T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}$ of T separately:

$$X(T)_{\text{in the coordinate system } x^i} := X^i\frac{\partial T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}}{\partial x^i}\partial_{x^{i_1}}\otimes\ldots\otimes\partial_{x^{i_k}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\,. \tag{A.9.2}$$

The resulting object does, however, *not* behave as a tensor under coordinate transformations, in the sense that the above form of the right-hand side will *not* be preserved under coordinate transformations: as an example, consider the one-form $T = dx$ on $\mathbb{R}^n$, which has vanishing derivative as defined by (A.9.2). When expressed in spherical coordinates we have

$$T = d(\rho\cos\varphi) = -\rho\sin\varphi d\varphi + \cos\varphi d\rho\,,$$

the partial derivatives of which are non-zero (both with respect to the original cartesian coordinates (x,y) and to the new spherical ones (ρ,φ)).

The Lie derivative $\mathscr{L}_X$ of Section A.8 maps tensors to tensors but does not resolve this question, because it is *not* linear under multiplication of X by a function.

The notion of *covariant derivative*, sometimes also referred to as *connection*, is introduced precisely to obtain a notion of derivative which has tensorial properties. By definition, a covariant derivative is a map which to a vector field X and a tensor field T assigns a tensor field of the same type as T, denoted by $\nabla_X T$, with the following properties:

1. $\nabla_X T$ is linear with respect to addition both with respect to X and T:

$$\nabla_{X+Y}T = \nabla_X T + \nabla_Y T\,, \qquad \nabla_X(T+Y) = \nabla_X T + \nabla_X Y\,; \tag{A.9.3}$$

2. $\nabla_X T$ is linear with respect to multiplication of X by functions f,

$$\nabla_{fX}T = f\nabla_X T\,; \tag{A.9.4}$$

3. and, finally, $\nabla_X T$ satisfies the *Leibniz rule* under multiplication of T by a differentiable function f:

$$\nabla_X(fT) = f\nabla_X T + X(f)T\,. \tag{A.9.5}$$

By definition, if T is a tensor field of rank (p,q), then for any vector field X the field $\nabla_X T$ is again a tensor of type (p,q). Since $\nabla_X T$ is linear in X, the field ∇T can naturally be viewed as a tensor field of rank $(p,q+1)$.

It is natural to ask whether covariant derivatives do exist at all in general and, if so, how many of them can there be. First, it immediately follows from the axioms above that if D and ∇ are two covariant derivatives, then

$$\Delta(X,T) := D_X T - \nabla_X T$$

is multi-linear both with respect to addition and multiplication by functions — the non-homogeneous terms $X(f)T$ in (A.9.5) cancel — and is thus a tensor field. Reciprocally, if ∇ is a covariant derivative and $\Delta(X,T)$ is bilinear with respect to addition and multiplication by functions, then

$$D_X T := \nabla_X T + \Delta(X,T) \tag{A.9.6}$$

is a new covariant derivative. So, at least locally, on tensors of valence (r,s) there are as many covariant derivatives as tensors of valence $(r+s, r+s+1)$.

We note that the sum of two covariant derivatives is *not* a covariant derivative. However, *convex* combinations of covariant derivatives, with coefficients which may vary from point to point, are again covariant derivatives. This remark allows one to construct covariant derivatives using partitions of unity: Let, indeed, $\{\mathscr{O}_i\}_{i\in\mathbb{N}}$ be an open covering of M by coordinate patches and let φ_i be an associated partition of unity. In each of those coordinate patches we can decompose a tensor field T as in (A.9.1), and define

$$D_X T := \sum_i \varphi_i X^j \partial_j (T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\,. \tag{A.9.7}$$

This procedure, which depends upon the choice of the coordinate patches and the choice of the partition of unity, defines *one* covariant derivative; all other covariant derivatives are then obtained from D using (A.9.6). Note that (A.9.2) is a special case of (A.9.7) when there exists a global coordinate system on M. Thus (A.9.2) *does* define a covariant derivative. However, the associated operation on tensor fields will *not* take the simple form (A.9.2) when we go to a different coordinate system $\{y^i\}$ in general.

A.9.1 Functions

The *canonical covariant derivative on functions* is defined as

$$\nabla_X(f) = X(f)\,,$$

and we will always use the above. This has all the right properties, so obviously covariant derivatives of functions exist. From what has been said, any covariant derivative on functions is of the form

$$\nabla_X f = X(f) + \alpha(X) f\,, \tag{A.9.8}$$

where α is a one-form. Conversely, given any one-form α, (A.9.8) defines a covariant derivative on functions. The addition of the lower-order term $\alpha(X)f$

(A.9.8) does not appear to be very useful here, but it turns out to be useful in geometric formulation of electrodynamics, or in *geometric quantization.* In any case such lower-order terms play an essential role when defining covariant derivatives of tensor fields.

A.9.2 Vectors

The simplest next possibility is that of a covariant derivative of vector fields. Let us not worry about existence at this stage, but assume that a covariant derivative exists, and work from there. (Anticipating, we will show shortly that a metric defines a covariant derivative, called the *Levi-Civita* covariant derivative, which is the unique covariant derivative operator satisfying a natural set of conditions, to be discussed below.)

We will first assume that we are working on a set $\Omega \subset M$ over which we have a *global trivialization* of the tangent bundle TM; by definition, this means that there exist vector fields e_a, $a = 1, \ldots, \dim M$, such that at every point $p \in \Omega$ the fields $e_a(p) \in T_pM$ form a basis of T_pM.[1]

Let θ^a denote the dual trivialization of T^*M — by definition the θ^a's satisfy

$$\boxed{\theta^a(e_b) = \delta^a_b}\,.$$

Given a covariant derivative ∇ on vector fields we set

$$\Gamma^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_b(X) e_a\,, \quad (A.9.9a)$$
$$\boxed{\Gamma^a{}_{bc} := \Gamma^a{}_b(e_c) = \theta^a(\nabla_{e_c} e_b)} \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_{bc} X^c e_a\,. \quad (A.9.9b)$$

The (locally defined) functions $\Gamma^a{}_{bc}$ are called *connection coefficients.* If $\{e_a\}$ is the coordinate basis $\{\partial_\mu\}$ we shall write

$$\Gamma^\mu{}_{\alpha\beta} := dx^\mu(\nabla_{\partial_\beta}\partial_\alpha) \quad \Big(\Longleftrightarrow \quad \nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\nu\mu}\partial_\sigma\Big)\,, \qquad (A.9.10)$$

etc. In this particular case the connection coefficients are usually called *Christoffel symbols.* We will sometimes write $\Gamma^\sigma_{\nu\mu}$ instead of $\Gamma^\sigma{}_{\nu\mu}$; note that the former convention is more common. By using the Leibniz rule (A.9.5) we find

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^a e_a) \\
&= X(Y^a)e_a + Y^a \nabla_X e_a \\
&= X(Y^a)e_a + Y^a \Gamma^b{}_a(X) e_b \\
&= (X(Y^a) + \Gamma^a{}_b(X) Y^b) e_a \\
&= (X(Y^a) + \Gamma^a{}_{bc} Y^b X^c) e_a\,, \qquad (A.9.11)
\end{aligned}$$

which gives various equivalent ways of writing $\nabla_X Y$. The (perhaps only locally defined) $\Gamma^a{}_b$'s are linear in X, and the collection $(\Gamma^a{}_b)_{a,b=1,\ldots,\dim M}$ is sometimes

[1] This is the case when Ω is a coordinate patch with coordinates (x^i), then the $\{e_a\}_{a=1,\ldots,\dim M}$ can be chosen to be equal to $\{\partial_i\}_{a=1,\ldots,\dim M}$. Recall that a manifold is said to be parallelizable if a basis of TM can be chosen globally over M — in such a case Ω can be taken equal to M. We emphasize that we are *not* assuming that M is parallelizable, so that equations such as (A.9.9) have only a local character in general.

referred to as the *connection one-form.* The one-covariant, one-contravariant tensor field ∇Y is defined as

$$\nabla Y := \nabla_a Y^b \theta^a \otimes e_b \Longleftrightarrow \nabla_a Y^b := \theta^b(\nabla_{e_a} Y) \Longleftrightarrow \boxed{\nabla_a Y^b = e_a(Y^b) + \Gamma^b{}_{ca} Y^c}. \tag{A.9.12}$$

We will often write ∇_a for ∇_{e_a}. Further, $\nabla_a Y^b$ will sometimes be written as $Y^b{}_{;a}$.

A.9.3 Transformation law

Consider a coordinate basis ∂_{x^i}, it is natural to enquire about the transformation law of the connection coefficients $\Gamma^i{}_{jk}$ under a change of coordinates $x^i \to y^k(x^i)$. To make things clear, let us write $\Gamma^i{}_{jk}$ for the connection coefficients in the x–coordinates, and $\hat\Gamma^i{}_{jk}$ for the ones in the y–cordinates. We calculate:

$$\begin{aligned}
\Gamma^i{}_{jk} &:= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\Big) \\
&= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\Big) \\
&= dx^i\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j} + \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\hat\Gamma^s{}_{\ell r}\,.
\end{aligned} \tag{A.9.13}$$

Summarising,

$$\boxed{\Gamma^i{}_{jk} = \hat\Gamma^s{}_{\ell r}\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k} + \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j}}\,. \tag{A.9.14}$$

Thus, the $\Gamma^i{}_{jk}$'s do *not* form a tensor; instead they transform as a tensor *plus* a non-homogeneous term containing second derivatives, as seen above.

EXERCICE A.9.1 Let $\Gamma^i{}_{jk}$ transform as in (A.9.14) under coordinate transformations. If X and Y are vector fields, define in local coordinates

$$\nabla_X Y := \Big(X(Y^i) + \Gamma^i{}_{jk} X^k Y^k\Big)\,\partial_i\,. \tag{A.9.15}$$

Show that $\nabla_X Y$ transforms as a vector field under coordinate transformations (and thus is a vector field). Hence, a collection of fields $\{\Gamma^i{}_{jk}\}$ satisfying the transformation law (A.9.14) can be used to define a covariant derivative using (A.9.15).

A.9.4 Torsion

Because the inhomogeneous term in (A.9.14) is symmetric under the interchange of i and j, it follows from (A.9.14) that

$$T^i_{jk} := \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

does transform as a tensor, called *the torsion tensor* of ∇.

An index-free definition of torsion proceeds as follows: Let ∇ be a covariant derivative defined for vector fields, the *torsion tensor* T is defined by the formula

$$\boxed{T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]}\,, \tag{A.9.16}$$

where $[X,Y]$ is the Lie bracket. We obviously have

$$T(X,Y) = -T(Y,X)\,. \tag{A.9.17}$$

Let us check that T is actually a tensor field: multi-linearity with respect to addition is obvious. To check what happens under multiplication by functions, in view of (A.9.17) it is sufficient to do the calculation for the first slot of T. We then have

$$\begin{aligned} T(fX,Y) &= \nabla_{fX} Y - \nabla_Y(fX) - [fX,Y] \\ &= f\Big(\nabla_X Y - \nabla_Y X\Big) - Y(f)X - [fX,Y]\,. \end{aligned} \tag{A.9.18}$$

To work out the last commutator term we compute, for any function φ,

$$[fX,Y](\varphi) = fX(Y(\varphi)) - \underbrace{Y(fX(\varphi))}_{=Y(f)X(\varphi)+fY(X(\varphi))} = f[X,Y](\varphi) - Y(f)X(\varphi)\,,$$

hence

$$[fX,Y] = f[X,Y] - Y(f)X\,, \tag{A.9.19}$$

and the last term here cancels the undesirable second-to-last term in (A.9.18), as required.

In a coordinate basis ∂_μ we have $[\partial_\mu,\partial_\nu]=0$ and one finds from (A.9.10)

$$\boxed{T(\partial_\mu,\partial_\nu) = (\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})\partial_\sigma}\,, \tag{A.9.20}$$

which shows that T is determined by twice the antisymmetrization of the $\Gamma^\sigma{}_{\mu\nu}$'s over the lower indices. In particular that last antisymmetrization produces a tensor field.

A.9.5 Covectors

Suppose that we are given a covariant derivative on vector fields, there is a natural way of inducing a covariant derivative on one-forms by imposing the condition that *the duality operation be compatible with the Leibniz rule*: given two vector fields X and Y together with a field of one-forms α, one sets

$$\boxed{(\nabla_X\alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)}\,. \tag{A.9.21}$$

Let us, first, check that (A.9.21) indeed defines a field of one-forms. The linearity, in the Y variable, with respect to addition is obvious. Next, for any function f we have

$$\begin{aligned}(\nabla_X\alpha)(fY) &= X(\alpha(fY)) - \alpha(\nabla_X(fY)) \\ &= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y + f\nabla_X Y) \\ &= f(\nabla_X\alpha)(Y)\,,\end{aligned}$$

as should be the case for one-forms. Next, we need to check that ∇ defined by (A.9.21) does satisfy the remaining axioms imposed on covariant derivatives. Again multi-linearity with respect to addition is obvious, as well as linearity with respect to multiplication of X by a function. Finally,

$$\begin{aligned}\nabla_X(f\alpha)(Y) &= X(f\alpha(Y)) - f\alpha(\nabla_X Y) \\ &= X(f)\alpha(Y) + f(\nabla_X\alpha)(Y)\,,\end{aligned}$$

as desired.

The duality pairing

$$T^*_pM \times T_pM \ni (\alpha, X) \to \alpha(X) \in \mathbb{R}$$

is sometimes called *contraction.* As already pointed out, the operation ∇ on one-forms has been defined in (A.9.21) so as to satisfy the *Leibniz rule under duality pairing*:

$$X(\alpha(Y)) = (\nabla_X\alpha)(Y) + \alpha(\nabla_X Y)\,; \qquad \text{(A.9.22)}$$

this follows directly from (A.9.21). This should not be confused with the Leibniz rule under multiplication by functions, which is part of the definition of a covariant derivative, and therefore always holds. It should be kept in mind that (A.9.22) does not necessarily hold for all covariant derivatives: if ${}^v\nabla$ is some covariant derivative on vectors, and ${}^f\nabla$ is some covariant derivative on one-forms, in general one will have

$$X(\alpha(Y)) \neq ({}^f\nabla_X)\alpha(Y) + \alpha({}^v\nabla_X Y)\,.$$

Using the basis-expression (A.9.11) of $\nabla_X Y$ and the definition (A.9.21) we have

$$\nabla_X\alpha = X^a\nabla_a\alpha_b\,\theta^b\,,$$

with

$$\begin{aligned}\boxed{\nabla_a\alpha_b} &:= (\nabla_{e_a}\alpha)(e_b) \\ &= e_a(\alpha(e_b)) - \alpha(\nabla_{e_a}e_b) \\ &= \boxed{e_a(\alpha_b) - \Gamma^c{}_{ba}\alpha_c}\,.\end{aligned}$$

A.9.6 Higher order tensors

It should now be clear how to extend ∇ to tensors of arbitrary valence: if T is r covariant and s contravariant one sets

$$\begin{aligned}(\nabla_X T)(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) &:= X\Big(T(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)\Big)\\ &-T(\nabla_X X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,\nabla_X X_r,\alpha_1,\ldots\alpha_s)\\ &-T(X_1,\ldots,X_r,\nabla_X\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,X_r,\alpha_1,\ldots\nabla_X\alpha_s)\,.\end{aligned} \tag{A.9.23}$$

The verification that this defines a covariant derivative proceeds in a way identical to that for one-forms. In a basis we have

$$\nabla_X T = X^a \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s}\theta^{a_1}\otimes\ldots\otimes\theta^{a_r}\otimes e_{b_1}\otimes\ldots\otimes e_{b_s}\,,$$

and (A.9.23) gives

$$\begin{aligned}\nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} &:= (\nabla_{e_a}T)(e_{a_1},\ldots,e_{a_r},\theta^{b_1},\ldots,\theta^{b_s})\\ &= e_a(T_{a_1\ldots a_r}{}^{b_1\ldots b_s}) - \Gamma^c{}_{a_1 a}T_{c\ldots a_r}{}^{b_1\ldots b_s} - \ldots - \Gamma^c{}_{a_r a}T_{a_1\ldots c}{}^{b_1\ldots b_s}\\ &\quad +\Gamma^{b_1}{}_{ca}T_{a_1\ldots a_r}{}^{c\ldots b_s}+\ldots+\Gamma^{b_s}{}_{ca}T_{a_1\ldots a_r}{}^{b_1\ldots c}\,.\end{aligned} \tag{A.9.24}$$

Carrying over the last two lines of (A.9.23) to the left-hand side of that equation one obtains the Leibniz rule for ∇ under pairings of tensors with vectors or forms. It should be clear from (A.9.23) that ∇ defined by that equation is the *only covariant derivative which agrees with the original one on vectors, and which satisfies the Leibniz rule under the pairing operation.* We will only consider such covariant derivatives in this work.

A.10 The Levi-Civita connection

One of the fundamental results in pseudo-Riemannian geometry is that of the existence of a torsion-free connection which preserves the metric:

THEOREM A.10.1 *Let g be a two-covariant symmetric non-degenerate tensor field on a manifold M. Then there exists a unique connection ∇ such that*

1. $\nabla g = 0$,
2. *the torsion tensor T of ∇ vanishes.*

PROOF: Suppose, first, that a connection satisfying the above is given. By the Leibniz rule we then have, for any vector fields X, Y and Z,

$$0 = (\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y,\nabla_X Z)\,. \tag{A.10.1}$$

We rewrite the same equation applying cyclic permutations to X, Y, and Z, with a minus sign for the last equation:

$$\begin{aligned} g(\nabla_X Y,Z)+g(Y,\nabla_X Z) &= X(g(Y,Z))\,,\\ g(\nabla_Y Z,X)+g(Z,\nabla_Y X) &= Y(g(Z,X))\,,\\ -g(\nabla_Z X,Y)-g(X,\nabla_Z Y) &= -Z(g(X,Y))\,.\end{aligned} \tag{A.10.2}$$

As the torsion tensor vanishes, the sum of the left-hand sides of these equations can be manipulated as follows:

$$\begin{aligned}
&g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\
&= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\
&= g(2\nabla_X Y - [X,Y], Z) + g(Y, [X,Z]) + g(X, [Y,Z]) \\
&= 2g(\nabla_X Y, Z) - g([X,Y], Z) + g(Y, [X,Z]) + g(X, [Y,Z])\,.
\end{aligned}$$

This shows that the sum of the three equations (A.10.2) can be rewritten as

$$\begin{aligned}
2g(\nabla_X Y, Z) &= g([X,Y], Z) - g(Y, [X,Z]) - g(X, [Y,Z]) \\
&\quad + X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\,. \qquad \text{(A.10.3)}
\end{aligned}$$

Since Z is arbitrary and g is non-degenerate, the left-hand side of this equation determines the vector field $\nabla_X Y$ uniquely, and uniqueness of ∇ follows.

To prove existence, let $S(X,Y)(Z)$ be defined as one half of the right-hand side of (A.10.3),

$$\begin{aligned}
S(X,Y)(Z) &= \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\
&\quad + g(Z, [X,Y]) - g(Y, [X,Z]) - g(X, [Y,Z])\Big)\,. \qquad \text{(A.10.4)}
\end{aligned}$$

Clearly S is linear with respect to addition in all fields involved. Let us check that it is also linear with respect to multiplication of Z by a function:

$$\begin{aligned}
S(X,Y)(fZ) &= \frac{f}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\
&\quad + g(Z, [X,Y]) - g(Y, [X,Z]) - g(X, [Y,Z])\Big) \\
&\quad + \frac{1}{2}\Big(X(f)g(Y,Z) + Y(f)g(Z,X) - g(Y, X(f)Z) - g(X, Y(f)Z)\Big) \\
&= fS(X,Y)(Z)\,. \qquad \text{(A.10.5)}
\end{aligned}$$

Since g is non-degenerate, we conclude that there exists a unique vector field $W(X,Y)$ such that

$$S(X,Y)(Z) = g(W(X,Y), Z)\,.$$

One readily checks that the assignment

$$(X,Y) \to W(X,Y) =: \nabla_X Y$$

satisfies all the requirements imposed on a covariant derivative $\nabla_X Y$.

It is immediate from (A.10.3), which is equivalent to (A.10.4), that the connection ∇ so defined is torsion free: Indeed, the sum of all-but-first terms at the right-hand side of (A.10.3) is symmetric in (X,Y), and the first term is what is needed to produce the torsion tensor when removing from (A.10.3) its counterpart with X and Y interchanged.

Finally, one checks that ∇ is metric-compatible by inserting $\nabla_X Y$ and $\nabla_X Z$, as defined by (A.10.3), into (A.10.1). This concludes the proof. $\square$

INCIDENTALLY: Let us give an index-notation version of the above. Using the definition of $\nabla_i g_{jk}$ we have

$$0 = \nabla_i g_{jk} \equiv \partial_i g_{jk} - \Gamma^\ell{}_{ji} g_{\ell k} - \Gamma^\ell{}_{ki} g_{\ell j}\,; \tag{A.10.6}$$

here we have written $\Gamma^i{}_{jk}$ instead of Γ^i_{jk}, as is standard in the literature. We rewrite this equation making cyclic permutations of indices, and changing the overall sign:

$$0 = -\nabla_j g_{ki} \equiv -\partial_j g_{ki} + \Gamma^\ell{}_{kj} g_{\ell i} + \Gamma^\ell{}_{ij} g_{\ell k}\,.$$

$$0 = -\nabla_k g_{ij} \equiv -\partial_k g_{ij} + \Gamma^\ell{}_{ik} g_{\ell j} + \Gamma^\ell{}_{jk} g_{\ell i}\,.$$

Adding the three equations and using symmetry of Γ^k_{ji} in ij one obtains

$$0 = \partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij} + 2\Gamma^\ell{}_{jk} g_{\ell i}\,,$$

Multiplying by g^{im} we obtain

$$\Gamma^m{}_{jk} = g^{mi}\Gamma^\ell{}_{jk} g_{\ell i} = \frac{1}{2} g^{mi}(\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk})\,. \tag{A.10.7}$$

This proves uniqueness.

A straightforward, though somewhat lengthy, calculation shows that the $\Gamma^m{}_{jk}$'s defined by (A.10.7) satisfy the transformation law (A.9.14). Exercice A.9.1 shows that the formula (A.9.15) defines a torsion-free connection. It then remains to check that the insertion of the $\Gamma^m{}_{jk}$'s, as given by (A.10.7), into the right-hand side of (A.10.6), indeed gives zero, proving existence. □

Let us check that (A.10.3) reproduces (A.10.7): Consider (A.10.3) with $X = \partial_\gamma$, $Y = \partial_\beta$ and $Z = \partial_\sigma$,

$$\begin{aligned} 2g(\nabla_\gamma \partial_\beta, \partial_\sigma) &= 2g(\Gamma^\rho{}_{\beta\gamma}\partial_\rho, \partial_\sigma) \\ &= 2g_{\rho\sigma}\Gamma^\rho{}_{\beta\gamma} \\ &= \partial_\gamma g_{\beta\sigma} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \end{aligned} \tag{A.10.8}$$

Multiplying this equation by $g^{\alpha\sigma}/2$ we then obtain

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\{\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\}}\,. \tag{A.10.9}$$

□

A.10.1 Geodesics and Christoffel symbols

A geodesic can be defined as the stationary point of the action

$$I(\gamma) = \int_a^b \underbrace{\frac{1}{2} g(\dot\gamma, \dot\gamma)(s)}_{=:\mathscr{L}(\gamma,\dot\gamma)}\, ds\,, \tag{A.10.10}$$

where $\gamma : [a,b] \to M$ is a differentiable curve. Thus,

$$\mathscr{L}(x^\mu, \dot x^\nu) = \frac{1}{2} g_{\alpha\beta}(x^\mu)\dot x^\alpha \dot x^\beta\,.$$

One readily finds the Euler-Lagrange equations for this Lagrange function:

$$\frac{d}{ds}\left(\frac{\partial\mathscr{L}}{\partial \dot x^\mu}\right) = \frac{\partial \mathscr{L}}{\partial x^\mu} \quad\Longleftrightarrow\quad \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,. \tag{A.10.11}$$

This provides a very convenient way of calculating the Christoffel symbols: given a metric g, write down $\mathscr{L}$, work out the Euler-Lagrange equations, and identify the Christoffels as the coefficients of the first derivative terms in those equations.

EXERCICE A.10.3 Prove (A.10.11). □

(The Euler-Lagrange equations for (A.10.10) are identical with those of

$$\tilde{I}(\gamma) = \int_a^b \sqrt{|g(\dot\gamma,\dot\gamma)(s)|}ds\,, \tag{A.10.12}$$

but (A.10.10) is more convenient to work with. For example, $\mathscr{L}$ is differentiable at points where $\dot\gamma$ vanishes, while $\sqrt{|g(\dot\gamma,\dot\gamma)(s)|}$ is not. The aesthetic advantage of (A.10.12), of being reparameterization-invariant, is more than compensated by the calculational convenience of $\mathscr{L}$.)

INCIDENTALLY: EXAMPLE A.10.5 As an example, consider a metric of the form

$$g = dr^2 + f(r)d\varphi^2\,.$$

Special cases of this metric include the Euclidean metric on $\mathbb{R}^2$ (then $f(r) = r^2$), and the canonical metric on a sphere (then $f(r) = \sin^2 r$, with r actually being the polar angle θ). The Lagrangian (A.10.12) is thus

$$L = \frac{1}{2}\left(\dot r^2 + f(r)\dot\varphi^2\right)\,.$$

The Euler-Lagrange equations read

$$\underbrace{\frac{\partial L}{\partial \varphi}}_{0} = \frac{d}{ds}\left(\frac{\partial L}{\partial\dot\varphi}\right) = \frac{d}{ds}(f(r)\dot\varphi)\,,$$

so that

$$0 = f\ddot\varphi + f'\dot r\dot\varphi = f\left(\ddot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 + 2\Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{rr}\dot r^2\right) \quad\Longrightarrow\quad \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{rr} = 0\,,\quad \Gamma^\varphi_{r\varphi} = \frac{f'}{2f}\,.$$

Similarly

$$\underbrace{\frac{\partial L}{\partial r}}_{f'\dot\varphi^2/2} = \frac{d}{ds}\left(\frac{\partial L}{\partial\dot r}\right) = \ddot r\,,$$

so that

$$\Gamma^r_{r\varphi} = \Gamma^r_{rr} = 0\,,\quad \Gamma^r_{\varphi\varphi} = -\frac{f'}{2}\,.$$

□

A.11 "Local inertial coordinates"

PROPOSITION A.11.1 *1. Let g be a Lorentzian metric, for every $p \in M$ there exists a neighborhood thereof with a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu} =$ $\mathrm{diag}(1,-1,\cdots,-1)$ at p.*

2. If g is differentiable, then the coordinates can be further chosen so that

$$\partial_\sigma g_{\alpha\beta} = 0 \tag{A.11.1}$$

at p.

The coordinates above will be referred to as *local inertial coordinates near* p.

REMARK A.11.2 An analogous result holds for any pseudo-Riemannian metric. Note that *normal coordinates*, constructed by shooting geodesics from p, satisfy the above. However, for metrics of finite differentiability, the introduction of normal coordinates leads to a loss of differentiability of the metric components, while the construction below preserves the order of differentiability.

PROOF: 1. Let y^μ be any coordinate system around p, shifting by a constant vector we can assume that p corresponds to $y^\mu = 0$. Let $e_a = e_a{}^\mu \partial/\partial y^\mu$ be any frame at p such that $g(e_a, e_b) = \eta_{ab}$ — such frames can be found by, *e.g.*, a Gram-Schmidt orthogonalisation. Calculating the determinant of both sides of the equation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}$$

we obtain, at p,

$$\det(g_{\mu\nu}) \det(e_a{}^\mu)^2 = -1\,,$$

which shows that $\det(e_a{}^\mu)$ is non-vanishing. It follows that the formula

$$y^\mu = e^\mu{}_a x^a$$

defines a (linear) diffeomorphism. In the new coordinates we have, again at p,

$$g\Big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\Big) = e^\mu{}_a e^\nu{}_b g\Big(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\Big) = \eta_{ab}\,. \tag{A.11.2}$$

2. We will use (A.9.14), which uses latin indices, so let us switch to that notation. Let x^i be the coordinates described in point 1., recall that p lies at the origin of those coordinates. The new coordinates $\hat{x}^j$ will be implicitly defined by the equations

$$x^i = \hat{x}^i + \frac{1}{2} A^i{}_{jk} \hat{x}^j \hat{x}^k\,,$$

where $A^i{}_{jk}$ is a set of constants, symmetric with respect to the interchange of j and k. Recall (A.9.14),

$$\hat{\Gamma}^i{}_{jk} = \Gamma^s{}_{\ell r} \frac{\partial \hat{x}^i}{\partial x^s} \frac{\partial x^\ell}{\partial \hat{x}^j} \frac{\partial x^r}{\partial \hat{x}^k} + \frac{\partial \hat{x}^i}{\partial x^s} \frac{\partial^2 x^s}{\partial \hat{x}^k \partial \hat{x}^j}\,; \tag{A.11.3}$$

here we use $\hat{\Gamma}^s{}_{\ell r}$ to denote the Christoffel symbols of the metric in the hatted coordinates. Then, at $x^i = 0$, this equation reads

$$\begin{aligned} \hat{\Gamma}^i{}_{jk} &= \Gamma^s{}_{\ell r} \underbrace{\frac{\partial \hat{x}^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial x^\ell}{\partial \hat{x}^j}}_{\delta^\ell_j} \underbrace{\frac{\partial x^r}{\partial \hat{x}^k}}_{\delta^r_k} + \underbrace{\frac{\partial x^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial^2 x^s}{\partial \hat{x}^k \partial \hat{x}^j}}_{A^s_{kj}} \\ &= \Gamma^i{}_{jk} + A^i{}_{kj}\,. \end{aligned}$$

Choosing A^i_{jk} as $-\Gamma^i{}_{jk}(0)$, the result follows.

INCIDENTALLY: If you do not like to remember formulae such as (A.9.14), proceed as follows: Let x^μ be the coordinates described in point 1. The new coordinates $\hat{x}^\alpha$ will be implicitly defined by the equations

$$x^\mu = \hat{x}^\mu + \frac{1}{2}A^\mu{}_{\alpha\beta}\hat{x}^\alpha\hat{x}^\beta \,,$$

where $A^\mu{}_{\alpha\beta}$ is a set of constants, symmetric with respect to the interchange of α and β. Set

$$\hat{g}_{\alpha\beta} := g\Big(\frac{\partial}{\partial\hat{x}^\alpha}, \frac{\partial}{\partial\hat{x}^\beta}\Big)\,, \qquad g_{\alpha\beta} := g\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big)\,.$$

Recall the transformation law

$$\hat{g}_{\mu\nu}(\hat{x}^\sigma) = g_{\alpha\beta}(x^\rho(\hat{x}^\sigma))\frac{\partial x^\alpha}{\partial\hat{x}^\mu}\frac{\partial x^\beta}{\partial\hat{x}^\nu}\,.$$

By differentiation one obtains at $x^\mu = \hat{x}^\mu = 0$,

$$\begin{aligned}\frac{\partial\hat{g}_{\mu\nu}}{\partial\hat{x}^\rho}(0) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + g_{\alpha\beta}(0)\Big(A^\alpha{}_{\mu\rho}\delta^\beta_\nu + \delta^\alpha_\mu A^\beta{}_{\nu\rho}\Big)\\ &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + A_{\nu\mu\rho} + A_{\mu\nu\rho}\,, \end{aligned}\tag{A.11.4}$$

where

$$A_{\alpha\beta\gamma} := g_{\alpha\sigma}(0)A^\sigma{}_{\beta\gamma}\,.$$

It remains to show that we can choose $A^\sigma{}_{\beta\gamma}$ so that the left-hand side can be made to vanish at p. An explicit formula for $A_{\sigma\beta\gamma}$ can be obtained from (A.11.4) by a cyclic permutation calculation similar to that in (A.10.2). After raising the first index, the final result is

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left\{\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right\}(0)\,;$$

the reader may wish to check directly that this does indeed lead to a vanishing right-hand side of (A.11.4).

□

A.12 Curvature

Let ∇ be a covariant derivative defined for vector fields, the curvature tensor is defined by the formula

$$\boxed{R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z}\,,\tag{A.12.1}$$

where, as elsewhere, $[X,Y]$ is the Lie bracket defined in (A.3.6). We note the anti-symmetry

$$R(X,Y)Z = -R(Y,X)Z\,.\tag{A.12.2}$$

It turns out this defines a tensor. Multi-linearity with respect to addition is obvious, but multiplication by functions require more work.

First, we have (see (A.9.19))

$$\begin{aligned}R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z\\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \underbrace{\nabla_{f[X,Y]-Y(f)X}Z}_{=f\nabla_{[X,Y]}Z-Y(f)\nabla_X Z}\\ &= fR(X,Y)Z\,.\end{aligned}$$

INCIDENTALLY: The simplest proof of linearity in the last slot proceeds via an index calculation in adapted coordinates; so while we will do the elegant, index-free version shortly, let us do the ugly one first. We use the coordinate system of Proposition A.11.1 below, in which the first derivatives of the metric vanish at the prescribed point p:

$$\begin{aligned}\nabla_i\nabla_j Z^k &= \partial_i(\partial_j Z^k - \Gamma^k{}_{\ell j}Z^\ell) + \underbrace{0\times\nabla Z}_{\text{at } p} \\ &= \partial_i\partial_j Z^k - \partial_i\Gamma^k{}_{\ell j}Z^\ell \quad \text{at } p\,. \end{aligned} \tag{A.12.3}$$

Antisymmetrising in i and j, the terms involving the second derivatives of Z drop out, so the result is indeed linear in Z. So $\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k$ is a tensor field linear in Z, and therefore can be written as $R^k{}_{\ell ij}Z^\ell$.

Note that $\nabla_i\nabla_j Z^k$ is, by definition, the tensor field of first covariant derivatives of the tensor field $\nabla_j Z^k$, while (A.12.1) involves covariant derivatives of vector fields only, so the equivalence of both approaches requires a further argument. This is provided in the calculation below leading to (A.12.7). □

We continue with

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \Big\{\nabla_X\Big(Y(f)Z + f\nabla_Y Z\Big)\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad -[X,Y](f)Z - f\nabla_{[X,Y]}Z \\ &= \Big\{\underbrace{X(Y(f))Z}_{a} + \underbrace{Y(f)\nabla_X Z + X(f)\nabla_Y Z}_{b} + f\nabla_X\nabla_Y Z\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad - \underbrace{[X,Y](f)Z}_{c} - f\nabla_{[X,Y]}Z\,. \end{aligned}$$

Now, a together with its counterpart with X and Y interchanged cancel out with c, while b is symmetric with respect to X and Y and therefore cancels out with its counterpart with X and Y interchanged, leading to the desired equality

$$R(X,Y)(fZ) \;=\; fR(X,Y)Z\,.$$

In a coordinate basis $\{e_a\} = \{\partial_\mu\}$ we find[2] (recall that $[\partial_\mu,\partial_\nu]=0$)

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &:= \langle dx^\alpha, R(\partial_\gamma,\partial_\delta)\partial_\beta\rangle \\ &= \langle dx^\alpha, \nabla_\gamma\nabla_\delta\partial_\beta\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \nabla_\gamma(\Gamma^\sigma{}_{\beta\delta}\partial_\sigma)\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \partial_\gamma(\Gamma^\sigma{}_{\beta\delta})\partial_\sigma + \Gamma^\rho{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\partial_\rho\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \{\partial_\gamma\Gamma^\alpha{}_{\beta\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\} - \{\cdots\}_{\delta\leftrightarrow\gamma}\,, \end{aligned}$$

leading finally to

$$\boxed{\boxed{R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}}}\,. \tag{A.12.4}$$

[2]The reader is warned that certain authors use other sign conventions either for $R(X,Y)Z$, or for $R^\alpha{}_{\beta\gamma\delta}$, or both. A useful table that lists the sign conventions for a series of standard GR references can be found on the backside of the front cover of [208].

In a general frame some supplementary commutator terms will appear in the formula for $R^a{}_{bcd}$.

INCIDENTALLY: An alternative way of introducing the Riemann tensor proceeds as in [278]; here we assume for simplicity that ∇ is torsion-free, but a similar calculation applies in general:

PROPOSITION A.12.3 *Let ∇ be torsion-free. There exists a tensor field $R^d{}_{abc}$ of type $(1,3)$ such that*

$$\nabla_a \nabla_b X^d - \nabla_b \nabla_a X^d = R^d{}_{cab} X^c \,. \tag{A.12.5}$$

PROOF: We need to check that the derivatives of X cancel. Now,

$$\begin{aligned}
\nabla_a \nabla_b X^d &= \partial_a (\underbrace{\nabla_b X^d}_{\partial_b X^d + \Gamma^d{}_{be} X^e}) + \Gamma^d_{ac} \underbrace{\nabla_b X^c}_{\partial_b X^c + \Gamma^c_{be} X^e} - \Gamma^e_{ab} \nabla_e X^d \\
&= \underbrace{\partial_a \partial_b X^d}_{=:1_{ab}} + \partial_a \Gamma^d{}_{be} X^e + \underbrace{\Gamma^d_{be} \partial_a X^e}_{=:2_{ab}} + \underbrace{\Gamma^d_{ac} \partial_b X^c}_{=:3_{ab}} + \Gamma^d_{ac} \Gamma^c_{be} X^e - \underbrace{\Gamma^e_{ab} \nabla_e X^d}_{=:4_{ab}} \,.
\end{aligned}$$

If we subtract $\nabla_b \nabla_a X^d$, then

1. 1_{ab} is symmetric in a and b, so will cancel out; similarly for 4_{ab} because ∇ has been assumed to have no torsion;
2. 2_{ab} will cancel out with 3_{ba}; similarly 3_{ab} will cancel out with 2_{ba}.

So the left-hand side of (A.12.5) is indeed linear in X^e. Since it is a tensor, the right-hand side also is. Since X^e is arbitrary, we conclude that $R^d{}_{cab}$ is a tensor of the desired type. □

We note the following:

THEOREM A.12.4 *There exists a coordinate system in which the metric tensor field has vanishing second derivatives at p if and only if its Riemann tensor vanishes at p. Furthermore, there exists a coordinate system in which the metric tensor field has constant entries near p if and only if the Riemann tensor vanishes near p.*

PROOF: The condition is necessary, since Riem is a tensor. The sufficiency will be admitted. □

The calculation of the curvature tensor may be a very traumatic experience. There is one obvious case where things are painless, when all $g_{\mu\nu}$'s are constants: in this case the Christoffels vanish, and so does the curvature tensor. Metrics with the last property are called *flat*.

For more general metrics, one way out is to use symbolic computer algebra. This can, e.g., be done online on `http://grtensor.phy.queensu.ca/NewDemo`. MATHEMATICA packages to do this can be found at URL's `http://www.math.washington.edu/~lee/Ricci`, or `http://grtensor.phy.queensu.ca/NewDemo`, or `http://luth.obspm.fr/~luthier/Martin-Garcia/xAct`. This last package is least-user-friendly as of today, but is the most flexible, especially for more involved computations.

We also note an algorithm of Benenti [22] to calculate the curvature tensor, starting from the variational principle for geodesics, which avoids writing-out explicitly all the Christoffel coefficients.

INCIDENTALLY: EXAMPLE A.12.6 As an example less trivial than a metric with constant coefficients, consider the round two sphere, which we write in the form

$$g = d\theta^2 + e^{2f} d\varphi^2 \,, \qquad e^{2f} = \sin^2\theta \,.$$

As seen in Example A.10.5, the Christoffel symbols are easily founds from the Lagrangian for geodesics:

$$\mathscr{L} = \frac{1}{2}(\dot\theta^2 + e^{2f}\dot\varphi^2) \,.$$

The Euler-Lagrange equations give

$$\Gamma^\theta{}_{\varphi\varphi} = -f' e^{2f} \,, \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = f' \,,$$

with the remaining Christoffel symbols vanishing. Using the definition of the Riemann tensor we then immediately find

$$R^\varphi{}_{\theta\varphi\theta} = -f'' - (f')^2 = -e^{-f}(e^f)'' = 1 \,. \tag{A.12.6}$$

All remaining components of the Riemann tensor can be obtained from this one by raising and lowering of indices, together with the symmetry operations which we are about to describe. This leads to

$$R_{ab} = g_{ab} \,, \qquad R = 2 \,.$$

□

Equation (A.12.1) is most frequently used “upside-down”, not as a definition of the Riemann tensor, but as a tool for calculating what happens when one changes the order of covariant derivatives. Recall that for partial derivatives we have

$$\partial_\mu \partial_\nu Z^\sigma = \partial_\nu \partial_\mu Z^\sigma \,,$$

but this is not true in general if partial derivatives are replaced by covariant ones:

$$\nabla_\mu \nabla_\nu Z^\sigma \neq \nabla_\nu \nabla_\mu Z^\sigma \,.$$

To find the correct formula let us consider the tensor field S defined as

$$Y \longrightarrow S(Y) := \nabla_Y Z \,.$$

In local coordinates, S takes the form

$$S = \nabla_\mu Z^\nu \, dx^\mu \otimes \partial_\nu \,.$$

It follows from the Leibniz rule — or, equivalently, from the definitions in Section A.9 — that we have

$$\begin{aligned} (\nabla_X S)(Y) &= \nabla_X (S(Y)) - S(\nabla_X Y) \\ &= \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \,. \end{aligned}$$

The commutator of the derivatives can then be calculated as

$$\begin{aligned} (\nabla_X S)(Y) - (\nabla_Y S)(X) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &\quad + \nabla_{[X,Y]} Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z \\ &= R(X,Y)Z - \nabla_{T(X,Y)} Z \,. \end{aligned} \tag{A.12.7}$$

Writing ∇S in the usual form

$$\nabla S = \nabla_\sigma S_\mu{}^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu = \nabla_\sigma \nabla_\mu Z^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu \,,$$

we are thus led to

$$\nabla_\mu \nabla_\nu Z^\alpha - \nabla_\nu \nabla_\mu Z^\alpha = R^\alpha{}_{\sigma\mu\nu} Z^\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma Z^\alpha \,. \tag{A.12.8}$$

In the important case of vanishing torsion, the coordinate-component equivalent of (A.12.1) is thus

$$\boxed{\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\sigma\mu\nu} X^\sigma} \,. \tag{A.12.9}$$

An identical calculation gives, still for torsionless connections,

$$\nabla_\mu \nabla_\nu a_\alpha - \nabla_\nu \nabla_\mu a_\alpha = -R^\sigma{}_{\alpha\mu\nu} a_\sigma \,. \tag{A.12.10}$$

For a general tensor t and torsion-free connection each tensor index comes with a corresponding Riemann tensor term:

$$\begin{aligned} &\nabla_\mu \nabla_\nu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \nabla_\nu \nabla_\mu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} = \\ &\quad -R^\sigma{}_{\alpha_1 \mu\nu} t_{\sigma \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \ldots - R^\sigma{}_{\alpha_r \mu\nu} t_{\alpha_1 \ldots \sigma}{}^{\beta_1 \ldots \beta_s} \\ &\quad + R^{\beta_1}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\sigma \ldots \beta_s} + \ldots + R^{\beta_s}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \sigma} \,. \end{aligned} \tag{A.12.11}$$

A.12.1 Bianchi identities

We have already seen the anti-symmetry property of the Riemann tensor, which in the index notation corresponds to the equation

$$R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \,. \tag{A.12.12}$$

There are a few other identities satisfied by the Riemann tensor, we start with the *first Bianchi identity*. Let $A(X,Y,Z)$ be any expression depending upon three vector fields X, Y, Z which is antisymmetric in X and Y, we set

$$\sum_{[XYZ]} A(X,Y,Z) := A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) \,, \tag{A.12.13}$$

thus $\sum_{[XYZ]}$ is a sum over cyclic permutations of the vectors X, Y, Z. Clearly,

$$\sum_{[XYZ]} A(X,Y,Z) = \sum_{[XYZ]} A(Y,Z,X) = \sum_{[XYZ]} A(Z,X,Y) \,. \tag{A.12.14}$$

Suppose, first, that X, Y and Z commute. Using (A.12.14) together with the definition (A.9.16) of the torsion tensor T we calculate

$$\begin{aligned} \sum_{[XYZ]} R(X,Y)Z &= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \Big) \\ &= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \quad \underbrace{(\nabla_Z X + T(X,Z))}_{\text{we have used } [X,Z]=0, \text{ see (A.9.16)}} \quad \Big) \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\sum_{[XYZ]} \nabla_X \nabla_Y Z - \sum_{[XYZ]} \nabla_Y \nabla_Z X}_{=0 \text{ (see (A.12.14))}} - \sum_{[XYZ]} \nabla_Y (\underbrace{T(X,Z)}_{=-T(Z,X)}) \\
&= \sum_{[XYZ]} \nabla_X (T(Y,Z))\,,
\end{aligned}
$$

and in the last step we have again used (A.12.14). This can be somewhat rearranged by using the definition of the covariant derivative of a higher order tensor (compare (A.9.23)) — equivalently, using the Leibniz rule rewritten upside-down:

$$(\nabla_X T)(Y,Z) = \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z)\,.$$

This leads to

$$
\begin{aligned}
\sum_{[XYZ]} \nabla_X (T(Y,Z)) &= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(\nabla_X Y, Z) + T(Y, \underbrace{\nabla_X Z}_{=T(X,Z)+\nabla_Z X}) \Big) \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) - T(\underbrace{T(X,Z)}_{=-T(Z,X)}, Y) \Big) \\
&\quad \underbrace{+ \sum_{[XYZ]} T(\nabla_X Y, Z) + \sum_{[XYZ]} \underbrace{T(Y, \nabla_Z X)}_{=-T(\nabla_Z X, Y)}}_{=0 \text{ (see (A.12.14))}} \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big)\,.
\end{aligned}
$$

Summarizing, we have obtained the first Bianchi identity:

$$\sum_{[XYZ]} R(X,Y)Z = \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big)\,, \tag{A.12.15}$$

under the hypothesis that X, Y and Z commute. However, both sides of this equation are tensorial with respect to X, Y and Z, so that they remain correct without the commutation hypothesis.

We are mostly interested in connections with vanishing torsion, in which case (A.12.15) can be rewritten as

$$\boxed{R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\gamma\delta\beta} + R^\alpha{}_{\delta\beta\gamma} = 0}\,. \tag{A.12.16}$$

Equivalently,

$$R^\alpha{}_{[\beta\gamma\delta]} = 0\,, \tag{A.12.17}$$

where brackets over indices denote complete antisymmetrisation, e.g.

$$
\begin{aligned}
A_{[\alpha\beta]} &= \tfrac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})\,, \\
A_{[\alpha\beta\gamma]} &= \tfrac{1}{6}(A_{\alpha\beta\gamma} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha} + A_{\alpha\gamma\beta} - A_{\beta\gamma\alpha})\,,
\end{aligned}
$$

etc.

Our next goal is the *second Bianchi identity*. We consider four vector fields X, Y, Z and W and we assume again that everybody commutes with everybody else. We calculate

$$\begin{aligned}
\sum_{[XYZ]} \nabla_X(R(Y,Z)W) &= \sum_{[XYZ]} \Big(\underbrace{\nabla_X\nabla_Y\nabla_Z W}_{=R(X,Y)\nabla_Z W+\nabla_Y\nabla_X\nabla_Z W} -\nabla_X\nabla_Z\nabla_Y W\Big) \\
&= \sum_{[XYZ]} R(X,Y)\nabla_Z W \\
&\quad \underbrace{+\sum_{[XYZ]} \nabla_Y\nabla_X\nabla_Z W - \sum_{[XYZ]} \nabla_X\nabla_Z\nabla_Y W}_{=0} \\
&\quad .
\end{aligned} \tag{A.12.18}$$

Next,

$$\begin{aligned}
\sum_{[XYZ]} (\nabla_X R)(Y,Z)W &= \sum_{[XYZ]} \Big(\nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W \\
&\quad -R(Y, \underbrace{\nabla_X Z}_{=\nabla_Z X+T(X,Z)})W - R(Y,Z)\nabla_X W\Big) \\
&= \sum_{[XYZ]} \nabla_X(R(Y,Z)W) \\
&\quad \underbrace{-\sum_{[XYZ]} R(\nabla_X Y,Z)W - \sum_{[XYZ]} \underbrace{R(Y,\nabla_Z X)W}_{=-R(\nabla_Z X,Y)W}}_{=0} \\
&\quad -\sum_{[XYZ]} \Big(R(Y,T(X,Z))W + R(Y,Z)\nabla_X W\Big) \\
&= \sum_{[XYZ]} \Big(\nabla_X(R(Y,Z)W) - R(T(X,Y),Z)W - R(Y,Z)\nabla_X W\Big) .
\end{aligned}$$

It follows now from (A.12.18) that the first term cancels out the third one, leading to

$$\sum_{[XYZ]} (\nabla_X R)(Y,Z)W = -\sum_{[XYZ]} R(T(X,Y),Z)W\,, \tag{A.12.19}$$

which is the desired second Bianchi identity for commuting vector fields. As before, because both sides are multi-linear with respect to addition and multiplication by functions, the result remains valid for arbitrary vector fields.

For torsionless connections the components equivalent of (A.12.19) reads

$$\boxed{R^\alpha{}_{\mu\beta\gamma;\delta} + R^\alpha{}_{\mu\gamma\delta;\beta} + R^\alpha{}_{\mu\delta\beta;\gamma} = 0}\,. \tag{A.12.20}$$

INCIDENTALLY: In the case of the Levi-Civita connection, the proof of the second Bianchi identity is simplest in coordinates in which the derivatives of the metric vanish at p: Indeed, a calculation very similar to the one leading to (A.12.25) below gives

$$\begin{aligned}\nabla_\delta R_{\alpha\mu\beta\gamma}(0) = \partial_\delta R_{\alpha\mu\beta\gamma}(0) = \\ \frac{1}{2}\Big\{\partial_\delta\partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\delta\partial_\beta\partial_\alpha g_{\mu\gamma} - \partial_\delta\partial_\gamma\partial_\mu g_{\alpha\beta} + \partial_\delta\partial_\gamma\partial_\alpha g_{\mu\beta}\Big\}(0)\,. \quad \text{(A.12.21)}\end{aligned}$$

and (A.12.20) follows by inspection □

A.12.2 Pair interchange symmetry

There is one more identity satisfied by the curvature tensor which is specific to the curvature tensor associated with the Levi-Civita connection, namely

$$g(X, R(Y,Z)W) = g(Y, R(X,W)Z)\,. \quad \text{(A.12.22)}$$

If one sets

$$\boxed{R_{abcd} := g_{ae}R^e{}_{bcd}}\,, \quad \text{(A.12.23)}$$

then (A.12.22) is equivalent to

$$\boxed{R_{abcd} = R_{cdab}}\,. \quad \text{(A.12.24)}$$

We will present two proofs of (A.12.22). The first is direct, but not very elegant. The second is prettier, but less insightful.

For the ugly proof, we suppose that the metric is twice-differentiable. By point 2. of Proposition A.11.1, in a neighborhood of any point $p \in M$ there exists a coordinate system in which the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ vanish at p. Equation (A.12.4) evaluated at p therefore reads

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} \\ &= \frac{1}{2}\Big\{g^{\alpha\sigma}\partial_\gamma(\partial_\delta g_{\sigma\beta} + \partial_\beta g_{\sigma\delta} - \partial_\sigma g_{\beta\delta}) \\ &\quad -g^{\alpha\sigma}\partial_\delta(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})\Big\} \\ &= \frac{1}{2}g^{\alpha\sigma}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}\,. \end{aligned}$$

Equivalently,

$$R_{\sigma\beta\gamma\delta}(0) = \frac{1}{2}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}(0)\,. \quad \text{(A.12.25)}$$

This last expression is obviously symmetric under the exchange of $\sigma\beta$ with $\gamma\delta$, leading to (A.12.24).

The above calculation traces back the pair-interchange symmetry to the definition of the Levi-Civita connection in terms of the metric tensor. As already mentioned, there exists a more elegant proof, where the origin of the symmetry is perhaps somewhat less apparent, which proceeds as follows: We start by noting that

$$0 = \nabla_a\nabla_b g_{cd} - \nabla_b\nabla_a g_{cd} = -R^e{}_{cab}g_{ed} - R^e{}_{dab}g_{ce}\,, \quad \text{(A.12.26)}$$

leading to anti-symmetry in the first two indices:

$$R_{abcd} = -R_{bacd}\,.$$

Next, using the cyclic symmetry for a torsion-free connection, we have

$$\begin{aligned}
R_{abcd} + R_{cabd} + R_{bcad} &= 0\,, \\
R_{bcda} + R_{dbca} + R_{cdba} &= 0\,, \\
R_{cdab} + R_{acdb} + R_{dacb} &= 0\,, \\
R_{dabc} + R_{bdac} + R_{abdc} &= 0\,.
\end{aligned} \tag{A.12.27}$$

The desired equation (A.12.24) follows now by adding the first two and subtracting the last two equations, using (A.12.26).

Remark A.12.8 In dimension two, the pair-interchange symmetry and the anti-symmetry in the last two indices immediately imply that the only non-zero components of the Riemann tensor are

$$R_{1212} = -R_{2112} = R_{2121} = -R_{2112}\,.$$

This is equivalent to the formula

$$R_{abcd} = \frac{R}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})\,,$$

as easily checked at a point p in a coordinate system where g_{ab} is diagonal at p.

In dimension three, a similar argument gives

$$R_{abcd} = (P_{ac}g_{bd} - P_{ad}g_{bc} + g_{ac}P_{bd} - g_{ad}P_{bc})\,, \tag{A.12.28}$$

where

$$P_{ab} := R_{ab} - \frac{R}{2}g_{ab}\,.$$

□

Incidentally: It is natural to enquire about the number of independent components of a tensor with the symmetries of a metric Riemann tensor in dimension n, the calculation proceeds as follows: as R_{abcd} is symmetric under the exchange of ab with cd, and anti-symmetric in each of these pairs, we can view it as a symmetric map from the space of anti-symmetric tensor with two indices. Now, the space of anti-symmetric tensors is $N = n(n-1)/2$ dimensional, while the space of symmetric maps in dimension N is $N(N+1)/2$ dimensional, so we obtain at most

$$\frac{n(n-1)(n^2-n+2)}{8}$$

free parameters. However, we need to take into account the cyclic identity:

$$R_{dabc} + R_{dbca} + R_{dcab} = 0\,. \tag{A.12.29}$$

If $a = b$ this reads

$$R_{daac} + R_{daca} + R_{dcaa} = 0\,,$$

which has already been accounted for. Similarly if $a = d$ we obtain

$$R_{abca} + R_{bcaa} + R_{caba} = 0\,,$$

which holds in view of the previous identities. We conclude that the only new identities which could possibly arise are those where *abcd* are all distinct. (Another way to see this is to note the identity

$$R_{a[bcd]} = R_{[abcd]} \,, \tag{A.12.30}$$

which holds for any tensor satisfying

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab} \,, \tag{A.12.31}$$

and which can be proved by writing explicitly all the terms in $R_{[abcd]}$; this is the same as adding the left-hand sides of the first and third equations in (A.12.27), and removing those of the second and fourth.)

Clearly no identity involving four distinct components of the Riemann tensor can be obtained using (A.12.31), so for each distinct set of four indices the Bianchi identity provides a constraint which is independent of (A.12.31). In dimension four (A.12.29) provides thus four candidate equations for another constraint, labeled by d, but it is easily checked that they all coincide either directly, or using (A.12.30). This leads to 20 free parameters at each space point. (Strictly speaking, to prove this one would still need to show that there are no further algebraic identities satisfied by the Riemann tensor, which is indeed the case.

Note that (A.12.30) shows that in dimension $n \geq 4$ the Bianchi identity introduces $\left(\begin{array}{c} n \\ 4 \end{array}\right)$ new constraints, leading to

$$\frac{n(n-1)(n^2-n+2)}{8} - \frac{n(n-1)(n-2)(n-3)}{12} = \frac{n^2(n^2-1)}{12} \tag{A.12.32}$$

independent components at each point. □

A.12.3 Summmary for the Levi-Civita connection

Here is a full list of algebraic symmetries of the curvature tensor of the Levi-Civita connection:

1. directly from the definition, we obtain

$$R^\delta{}_{\gamma\alpha\beta} = -R^\delta{}_{\gamma\beta\alpha} \,; \tag{A.12.33}$$

2. the next symmetry, known as the *first Bianchi identity*, is less obvious:

$$R^\delta{}_{\gamma\alpha\beta} + R^\delta{}_{\alpha\beta\gamma} + R^\delta{}_{\beta\gamma\alpha} = 0 \quad \Longleftrightarrow \quad R^\delta{}_{[\gamma\alpha\beta]} = 0 \,; \tag{A.12.34}$$

3. and finally we have the pair-interchange symmetry:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \,. \tag{A.12.35}$$

Here, of course, $R_{\gamma\delta\alpha\beta} = g_{\gamma\sigma} R^\sigma{}_{\delta\alpha\beta}$.

It is not obvious, but true, that the above exhaust the list of all independent algebraic identities satisfied by $R_{\alpha\beta\gamma\delta}$.

As a consequence of (A.12.33) and (A.12.35) we find

$$R_{\alpha\beta\delta\gamma} = R_{\delta\gamma\alpha\beta} = -R_{\delta\gamma\beta\alpha} = -R_{\beta\alpha\gamma\delta} \,,$$

and so the Riemann tensor is also anti-symmetric in its first two indices:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}\,. \tag{A.12.36}$$

The *Ricci tensor* is defined as

$$R_{\alpha\beta} := R^\sigma{}_{\alpha\sigma\beta}\,.$$

The pair-interchange symmetry implies that the Ricci tensor is symmetric:

$$R_{\alpha\beta} = g^{\sigma\rho}R_{\sigma\alpha\rho\beta} = g^{\sigma\rho}R_{\rho\beta\sigma\alpha} = R_{\beta\alpha}\,.$$

Finally we have the differential *second Bianchi identity*:

$$\nabla_\alpha R^\sigma{}_{\delta\beta\gamma} + \nabla_\beta R^\sigma{}_{\delta\gamma\alpha} + \nabla_\gamma R^\sigma{}_{\delta\alpha\beta} = 0 \quad\Longleftrightarrow\quad \nabla_{[\alpha}R_{\beta\gamma]\mu\nu} = 0\,. \tag{A.12.37}$$

A.12.4 Curvature of product metrics

Let (M,g) and (N,h) be two pseudo-Riemannian manifolds, on the product manifold $M\times N$ we define a metric $g\oplus h$ as follows: Every element of $T(M\times N)$ can be uniquely written as $X\oplus Y$ for some $X\in TM$ and $Y\in TN$. We set

$$(g\oplus h)(X\oplus Y,\hat X\oplus\hat Y) = g(X,\hat X) + h(Y,\hat Y)\,.$$

Let ∇ be the Levi-Civita connection associated with g, D that associated with h, and $\mathscr{D}$ the one associated with $g\oplus h$. To understand the structure of ∇, we note that sections of $T(M\times N)$ are linear combinations, with coefficients in $C^\infty(M\times N)$, of elements of the form $X\oplus Y$, where $X\in\Gamma(TM)$ and $Y\in\Gamma(TN)$. (Thus, X does not depend upon $q\in N$ and Y does not depend upon $p\in M$.) We claim that for such fields $X\oplus Y$ and $W\oplus Z$ we have

$$\mathscr{D}_{X\oplus Y}(W\oplus Z) = \nabla_X W\oplus D_Y Z\,. \tag{A.12.38}$$

(If true, (A.12.38) together with the Leibniz rule characterises $\mathscr{D}$ uniquely.) To verify (A.12.38), we check first that $\mathscr{D}$ has no torsion:

$$\begin{aligned}
\mathscr{D}_{X\oplus Y}(W\oplus Z) - \mathscr{D}_{W\oplus Z}(X\oplus Y) &= \nabla_X W\oplus D_Y Z - \nabla_W X\oplus D_Z Y\\
&= (\nabla_X W - \nabla_W X)\oplus(D_Y Z - D_Z Y)\\
&= [X,W]\oplus[Y,Z]\\
&= [X\oplus Y, W\oplus Z]\,.
\end{aligned}$$

(In the last step we have used $[X\oplus 0, 0\oplus Z] = [0\oplus Y, W\oplus 0] = 0$.) Next, we check metric compatibility:

$$\begin{aligned}
&X\oplus Y\Big((g\oplus h)(W\oplus Z,\hat W\oplus\hat Z)\Big)\\
&\quad = X\oplus Y\Big(g(W,\hat W) + h(Z,\hat Z)\Big)\\
&\quad = \underbrace{X\Big(g(W,\hat W)\Big)}_{g(\nabla_X W,\hat W)+g(W,\nabla_X\hat W)} + \underbrace{Y\Big(h(Z,\hat Z)\Big)}_{h(D_Y Z,\hat Z)+h(Z,D_Y\hat Z)}\\
&\quad = \underbrace{g(\nabla_X W,\hat W) + h(D_Y Z,\hat Z)}_{(g\oplus h)(\nabla_X W\oplus D_Y Z,\hat W\oplus\hat Z)} + \underbrace{g(W,\nabla_X\hat W) + h(Z,D_Y\hat Z)}_{(g\oplus h)(W\oplus Z,\nabla_X\hat W\oplus D_Y\hat Z)}\\
&\quad = (g\oplus h)(\mathscr{D}_{X\oplus Y}W\oplus Z,\hat W\oplus\hat Z) + (g\oplus h)(W\oplus Z,\mathscr{D}_{X\oplus Y}\hat W\oplus\hat Z)\,.
\end{aligned}$$

Uniqueness of Levi-Civita connections proves (A.12.38).

Let Riem(k) denote the Riemann tensor of the metric k. It should be clear from (A.12.38) that the Riemann tensor of $g \oplus h$ has a sum structure,

$$\mathrm{Riem}(g \oplus h) = \mathrm{Riem}(g) \oplus \mathrm{Riem}(h)\,. \tag{A.12.39}$$

More precisely,

$$\mathrm{Riem}(g \oplus h)(X \oplus Y, \hat{X} \oplus \hat{Y})W \oplus Z = \mathrm{Riem}(g)(X, \hat{X})W \oplus \mathrm{Riem}(h)(Y, \hat{Y})Z\,. \tag{A.12.40}$$

This implies

$$\mathrm{Ric}(g \oplus h) = \mathrm{Ric}(g) \oplus \mathrm{Ric}(h)\,, \tag{A.12.41}$$

in the sense that

$$\mathrm{Ric}(g \oplus h)(X \oplus Y, \hat{X} \oplus \hat{Y}) = \mathrm{Ric}(g)(X, \hat{X}) \oplus \mathrm{Ric}(h)(Y, \hat{Y})\,, \tag{A.12.42}$$

and

$$\mathrm{tr}_{g\oplus h}\mathrm{Ric}(g \oplus h) \;\; = \;\; \mathrm{tr}_g\mathrm{Ric}(g) + \mathrm{tr}_h\mathrm{Ric}(h)\,. \tag{A.12.43}$$

A.12.5 An identity for the Riemann tensor

We write $\delta^{\alpha\beta}_{\gamma\delta}$ for $\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta} \equiv \frac{1}{2}(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta})$, etc.

For completeness we prove the following identity satisfied by the Riemann tensor, which is valid in any dimension, is clear in dimensions two and three, implies the double-dual identity for the Weyl tensor in dimension four, and is probably well known in higher dimensions as well:

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\gamma\delta} = \frac{1}{3!}\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu} R - 4\delta^{[\alpha}_{[\mu} R^{\beta]}{}_{\nu]}\right)\,. \tag{A.12.44}$$

The above holds for any tensor field satisfying

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\delta\gamma}\,. \tag{A.12.45}$$

To prove (A.12.44) one can calculate as follows:

$$\begin{aligned}
4!\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\gamma\delta} \;\; = \;\; & 2[\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho}\right) \\
& -\delta^{\alpha}_{\nu}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho}\right) \\
& +\delta^{\alpha}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right) \\
& -\delta^{\alpha}_{\sigma}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho} + \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right)]R^{\rho\sigma}{}_{\gamma\delta} \\
= \;\; & 2\left(2\delta^{\alpha\beta}_{\mu\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - 4\delta^{\alpha\gamma}_{\mu\nu}\delta^{\beta\delta}_{\rho\sigma} + 4\delta^{\beta\gamma}_{\mu\nu}\delta^{\alpha\delta}_{\rho\sigma} + 2\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right) R^{\rho\sigma}{}_{\gamma\delta} \\
= \;\; & 4\left(\delta^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}{}_{\gamma\delta} - 2\delta^{\alpha\gamma}_{\mu\nu} R^{\beta\sigma}{}_{\gamma\sigma} + 2\delta^{\beta\gamma}_{\mu\nu} R^{\alpha\sigma}{}_{\gamma\sigma} + R^{\alpha\beta}{}_{\mu\nu}\right) \\
= \;\; & 4\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}{}_{\gamma\delta} - 4\delta^{[\alpha}_{[\mu} R^{\beta]\gamma}{}_{\nu]\gamma}\right)\,.
\end{aligned} \tag{A.12.46}$$

If the sums are over all indices we obtain (A.12.44). The reader is warned, however, that in some of our calculations the sums will be only over a subset of all possible indices, in which case the last equation remains valid but the last two terms in (A.12.46) *cannot* be replaced by the Ricci scalar and the Ricci tensor.

Let us show that the *double-dual identity* for the Weyl tensor does indeed follow from (A.12.44). For this, note that in spacetime dimension four we have

$$4!\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}\,, \tag{A.12.47}$$

since both sides are completely anti-symmetric in the upper and lower indices, and coincide when both pairs equal 0123. Hence, since the Weyl tensor $W^{\rho\sigma}{}_{\gamma\delta}$ has all the required symmetries and vanishing traces, we find

$$4W^{\alpha\beta}{}_{\mu\nu} \underbrace{=}_{\text{by (A.12.46)}} 4!\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta}\,. \tag{A.12.48}$$

This is equivalent to the desired identity

$$\epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta} = W^{\alpha\beta}{}_{\mu\nu}\,\epsilon_{\alpha\beta\gamma\delta}\,. \tag{A.12.49}$$

A.13 Geodesics

An *affinely parameterised geodesic* γ is a maximally extended solution of the equation

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

(compare (A.10.11).) It is a fundamental postulate of general relativity that physical observers move on timelike geodesics. This motivates the following definition: an *observer* is a maximally extended future directed timelike geodesics.

INCIDENTALLY: It is sometimes convenient to consider geodesics which are not necessarily affinely parameterised. Those are solutions of

$$\nabla_{\frac{d\gamma}{d\lambda}}\frac{d\gamma}{d\lambda} = \chi\frac{d\gamma}{d\lambda}\,. \tag{A.13.1}$$

Indeed, let us show that a change of parameter obtained by solving the equation

$$\frac{d^2\lambda}{ds^2} + \chi\left(\frac{d\lambda}{ds}\right)^2 = 0 \tag{A.13.2}$$

brings (A.13.2) to the form (A.13.2): under a change of parameter $\lambda = \lambda(s)$ we have

$$\frac{d\gamma^\mu}{ds} = \frac{d\lambda}{ds}\frac{d\gamma^\mu}{d\lambda}\,,$$

and

$$\begin{aligned}
\frac{D}{ds}\frac{d\gamma^\nu}{ds} &= \frac{D}{ds}(\frac{d\lambda}{ds}\frac{d\gamma^\nu}{d\lambda}) \\
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + \frac{d\lambda}{ds}\frac{D}{ds}\frac{d\gamma^\nu}{d\lambda} \\
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + (\frac{d\lambda}{ds})^2\frac{D}{d\lambda}\frac{d\gamma^\nu}{d\lambda} \\
&= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + (\frac{d\lambda}{ds})^2\chi\frac{d\gamma^\nu}{d\lambda}\,,
\end{aligned}$$

and the choice indicated above gives zero, as desired. □

Let f be a smooth function and let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; by definition, this means that $d\gamma^\mu/d\lambda = \nabla^\mu f$. The following provides a convenient tool for finding geodesics:

PROPOSITION A.13.2 (Integral curves of gradients) *Let f be a function satisfying*

$$g(\nabla f, \nabla f) = \psi(f)\,,$$

for some function ψ. Then the integral curves of ∇f are geodesics, affinely parameterised if $\psi' = 0$.

PROOF: We have

$$\dot\gamma^\alpha \nabla_\alpha \dot\gamma^\beta = \nabla^\alpha f \nabla_\alpha \nabla^\beta f = \nabla^\alpha f \nabla^\beta \nabla_\alpha f = \frac{1}{2}\nabla^\beta(\nabla^\alpha f \nabla_\alpha f) = \frac{1}{2}\nabla^\beta \psi(f) = \frac{1}{2}\psi' \nabla^\beta f\,. \tag{A.13.3}$$

Let λ the natural parameter on the integral curves of ∇f,

$$\frac{d\gamma^\mu}{d\lambda} = \nabla^\mu f\,,$$

then (A.13.3) can be rewritten as

$$\frac{D}{d\lambda}\frac{d\gamma^\mu}{d\lambda} = \frac{1}{2}\psi' \frac{d\gamma^\mu}{d\lambda}\,.$$

□

A significant special case is that of a coordinate function $f = x^i$. Then

$$g(\nabla f, \nabla f) = g(\nabla x^i, \nabla x^i) = g^{ii} \text{ (no summation)}\,.$$

For example, in Minkowski spacetime, all $g^{\mu\nu}$'s are constant, which shows that the integral curves of the gradient of any coordinate, and hence also of any linear combination of coordinates, are affinely parameterized geodesics. An other example is provided by the coordinate r in Schwarzschild spacetime, where $g^{rr} = 1 - 2m/r$; this is indeed a function of r, so the integral curves of $\nabla r = (1 - 2m/r)\partial_r$ are (non-affinely parameterized) geodesics.

Similarly one shows:

PROPOSITION A.13.3 *Suppose that $d(g(X,X)) = 0$ along an orbit γ of a Killing vector field X. Then γ is a geodesic.*

EXERCICE A.13.4 Consider the Killing vector field $X = \partial_t + \Omega\partial_\varphi$, where Ω is a constant, in the Schwarzschild spacetime. Find all geodesic orbits of X by studying the equation $d(g(X,X)) = 0$. □

A.14 Geodesic deviation (Jacobi equation)

Suppose that we have a one parameter family of geodesics

$$\gamma(s,\lambda) \text{ (in local coordinates, } (\gamma^\alpha(s,\lambda))),$$

where s is an affine parameter along the geodesic, and λ is a parameter which labels the geodesics. Set

$$Z(s,\lambda) := \frac{\partial\gamma(s,\lambda)}{\partial\lambda} \equiv \frac{\partial\gamma^\alpha(s,\lambda)}{\partial\lambda}\partial_\alpha\,;$$

for each λ this defines a vector field Z along $\gamma(s,\lambda)$, which measures how nearby geodesics deviate from each other, since, to first order, using a Taylor expansion,

$$\gamma^\alpha(s,\lambda) = \gamma^\alpha(s,\lambda_0) + Z^\alpha(\lambda-\lambda_0) + O((\lambda-\lambda_0)^2)\,.$$

To measure how a vector field W changes along $s \mapsto \gamma(s,\lambda)$, one introduces the differential operator D/ds, defined as

$$\begin{aligned} \frac{DW^\mu}{ds} &:= \frac{\partial(W^\mu\circ\gamma)}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha && \text{(A.14.1)}\\ &= \dot\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha && \text{(A.14.2)}\\ &= \dot\gamma^\beta\nabla_\beta W^\mu\,. && \text{(A.14.3)}\end{aligned}$$

(It would perhaps be more logical to write $\frac{DW^\mu}{\partial s}$ in the current context, but this is rarely done. Another notation for $\frac{D}{ds}$ often used in the mathematical literature is $\gamma_*\partial_s$.) The last two lines only make sense if W is defined in a whole neighbourhood of γ, but for the first it suffices that $W(s)$ be defined along $s\mapsto\gamma(s,\lambda)$. (One possible way of making sense of the last two lines is to extend, whenever possible, W^μ to any smooth vector field defined in a neighorhood of $\gamma^\mu(s,\lambda)$, and note that the result is independent of the particular choice of extension because the equation involves only derivatives tangential to $s\mapsto\gamma^\mu(s,\lambda)$.)

Analogously one sets

$$\begin{aligned} \frac{DW^\mu}{d\lambda} &:= \frac{\partial(W^\mu\circ\gamma)}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha && \text{(A.14.4)}\\ &= \partial_\lambda\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha && \text{(A.14.5)}\\ &= Z^\beta\nabla_\beta W^\mu\,. && \text{(A.14.6)}\end{aligned}$$

Note that since $s\to\gamma(s,\lambda)$ is a geodesic we have from (A.14.1) and (A.14.3)

$$\frac{D^2\gamma^\mu}{ds^2} := \frac{D\dot\gamma^\mu}{ds} = \frac{\partial^2\gamma^\mu}{\partial s^2} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\dot\gamma^\alpha = 0\,. \tag{A.14.7}$$

(This is sometimes written as $\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu = 0$, which is again an abuse of notation since typically we will only know $\dot\gamma^\mu$ as a function of s, and so there is no such thing as $\nabla_\alpha\dot\gamma^\mu$.) Furthermore,

$$\frac{DZ^\mu}{ds} \underbrace{=}_{\text{(A.14.1)}} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{\text{(A.14.4)}} \frac{D\dot\gamma^\mu}{d\lambda}\,, \tag{A.14.8}$$

(The abuse-of-notation derivation of the same formula proceeds as:

$$\begin{aligned}\nabla_{\dot\gamma} Z^\mu &= \dot\gamma^\nu \nabla_\nu Z^\mu = \dot\gamma^\nu \nabla_\nu \partial_\lambda \gamma^\mu \underbrace{=}_{(A.14.3)} \frac{\partial^2 \gamma^\mu}{\partial s \partial \lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta \partial_\lambda \gamma^\alpha \underbrace{=}_{(A.14.6)} Z^\beta \nabla_\beta \dot\gamma^\mu \\ &= \nabla_Z \dot\gamma^\mu \,, \end{aligned} \qquad (A.14.9)$$

which can then be written as

$$\nabla_{\dot\gamma} Z = \nabla_Z \dot\gamma \,.) \qquad (A.14.10)$$

We have the following identity for any vector field W defined along $\gamma^\mu(s,\lambda)$, which can be proved by e.g. repeating the calculation leading to (A.12.9):

$$\frac{D}{ds}\frac{D}{d\lambda} W^\mu - \frac{D}{d\lambda}\frac{D}{ds} W^\mu = R^\mu{}_{\delta\alpha\beta}\dot\gamma^\alpha Z^\beta W^\delta \,. \qquad (A.14.11)$$

If $W^\mu = \dot\gamma^\mu$ the second term at the left-hand side of (A.14.11) vanishes, and from $\frac{D}{d\lambda}\dot\gamma = \frac{D}{ds} Z$ we obtain

$$\frac{D^2 Z^\mu}{ds^2}(s) = R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\alpha Z^\beta \dot\gamma^\sigma \,. \qquad (A.14.12)$$

This is an equation known as the *Jacobi equation*, or as the *geodesic deviation equation*; in index-free notation:

$$\boxed{\frac{D^2 Z}{ds^2} = R(\dot\gamma, Z)\dot\gamma} \,. \qquad (A.14.13)$$

Solutions of (A.14.13) are called *Jacobi fields* along γ.

INCIDENTALLY: The advantage of the abuse-of-notation equations above is that, instead of adapting the calculation, one can directly invoke the result of Proposition A.12.3to obtain (A.14.11):

$$\begin{aligned}\frac{D^2 Z^\mu}{ds^2}(s) &= \dot\gamma^\alpha \nabla_\alpha (\dot\gamma^\beta \nabla_\beta Z^\mu) \\ &= \dot\gamma^\alpha \nabla_\alpha (Z^\beta \nabla_\beta \dot\gamma^\mu) \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha \nabla_\alpha \nabla_\beta \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\dot\gamma^\mu + Z^\beta \dot\gamma^\alpha \nabla_\beta \nabla_\alpha \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\sigma + Z^\beta \dot\gamma^\alpha \nabla_\beta \nabla_\alpha \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\sigma \\ &\quad + Z^\beta \nabla_\beta \underbrace{(\dot\gamma^\alpha \nabla_\alpha \dot\gamma^\mu)}_{0} - (Z^\beta \nabla_\beta \dot\gamma^\alpha)\nabla_\alpha \dot\gamma^\mu \,. \end{aligned} \qquad (A.14.14)$$

A renaming of indices in the first and the last term gives

$$(\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu - (Z^\beta \nabla_\beta \dot\gamma^\alpha)\nabla_\alpha \dot\gamma^\mu = (\dot\gamma^\alpha \nabla_\alpha Z^\beta - Z^\alpha \nabla_\alpha \dot\gamma^\beta)\nabla_\beta \dot\gamma^\mu \,,$$

which is zero by (A.14.10). This leads again to (A.14.12). □

A.15 Exterior algebra

A preferred class of tensors is provided by those that are totally antisymmetric in all indices. Such k-covariant tensors are called *k-forms*. They are of special interest because they can naturally be used for integration. Furthermore, on such tensors one can introduce a differentiation operation, called *exterior derivative*, which does not require a connection.

By definition, functions are *zero-forms*, and covectors are *one-forms*.

Let α_i, $i = 1, \ldots, k$, be a collection of one-forms, the *exterior product* of the α_i's is a k-form defined as

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \det\left(\alpha_i(X_j)\right), \tag{A.15.1}$$

where $\det\left(\alpha_i(X_j)\right)$ denotes the determinant of the matrix obtained by applying all the α_i's to all the vectors X_j. For example

$$(dx^a \wedge dx^b)(X, Y) = X^a Y^b - Y^a X^b .$$

Note that

$$dx^a \wedge dx^b = dx^a \otimes dx^b - dx^b \otimes dx^a ,$$

which is twice the antisymmetrisation $dx^{[a} \otimes dx^{b]}$.

Quite generally, if α is a totally anti-symmetric k-covariant tensor with coordinate coefficients $\alpha_{a_1 \ldots a_k}$, then

$$\begin{aligned} \alpha &= \alpha_{a_1 \ldots a_k} dx^{a_1} \otimes \cdots \otimes dx^{a_k} \\ &= \alpha_{a_1 \ldots a_k} dx^{[a_1} \otimes \cdots \otimes dx^{a_k]} \\ &= \frac{1}{k!} \alpha_{a_1 \ldots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} \\ &= \sum_{a_1 < \cdots < a_k} \alpha_{a_1 \ldots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} . \end{aligned} \tag{A.15.2}$$

The middle formulae exhibits the factorial coefficients needed to go from tensor components to the components in the $dx^{a_1} \wedge \cdots \wedge dx^{a_k}$ basis.

Equation (A.15.2) makes it clear that in dimension n for any non-trivial k-form we have $k \le n$. It also shows that the dimension of the space of k-forms, with $0 \le k \le n$, equals

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} .$$

A differential form is defined as a linear combination of k-forms, with k possibly taking different values for different summands.

Let Y be a vector and α a k-form. The *contraction* $Y \rfloor \alpha$, also called the *interior product* of Y and α, is a $(k-1)$-form defined as

$$(Y \rfloor \alpha)(X_1, \ldots, X_{k-1}) := \alpha(Y, X_1, \ldots, X_{k-1}) . \tag{A.15.3}$$

The operation $Y \rfloor$ is often denoted by i_Y.

Let α be a k-form and β an ℓ-form, the exterior product $\alpha\wedge\beta$ of α and β, also called *wedge product*, is defined using bilinearity:

$$\begin{aligned}
\alpha\wedge\beta \equiv\\
&\left(\sum_{a_1<\cdots<a_k}\alpha_{a_1\ldots a_k}dx^{a_1}\wedge\cdots\wedge dx^{a_k}\right)\wedge\left(\sum_{b_1<\cdots<b_\ell}\beta_{b_1\ldots b_\ell}dx^{b_1}\wedge\cdots\wedge dx^{b_\ell}\right)\\
&:= \sum_{a_1<\cdots<a_k\,,\ b_1<\cdots<b_\ell}\alpha_{a_1\ldots a_k}\beta_{b_1\ldots b_\ell}\times\\
&\qquad dx^{a_1}\wedge\cdots\wedge dx^{a_k}\wedge dx^{b_1}\wedge\cdots\wedge dx^{b_\ell}\,. \qquad \text{(A.15.4)}
\end{aligned}$$

The product so-defined is associative:

$$\alpha\wedge(\beta\wedge\gamma)=(\alpha\wedge\beta)\wedge\gamma=:\alpha\wedge\beta\wedge\gamma\,. \qquad \text{(A.15.5)}$$

Incidentally: In order to establish (A.15.5), we start by rewriting the definition of the wedge product of a k-form α and l-form β as

$$(\alpha\wedge\beta)\,(X_1\ldots,X_{k+l}):=\frac{1}{k!}\frac{1}{l!}\sum_{\pi\in S_{k+l}}\mathrm{sgn}(\pi)\,(\alpha\otimes\beta)\,(X_{\pi(1)},\ldots,X_{\pi(k+l)})\,, \qquad \text{(A.15.6)}$$

where $X_i\in\Gamma(TM)$ for $i=1,\ldots,k+l$.

Let S_p denote the group of permutations of p elements and let $\Omega^\ell(M)$ denote the space of ℓ-forms. For $\alpha\in\Omega^k(M)$, $\beta\in\Omega^l(M)$ and $\gamma\in\Omega^m(M)$ we have

$$\begin{aligned}
&((\alpha\wedge\beta)\wedge\gamma)(X_1,\ldots,X_{k+l+m})\\
&= \frac{1}{(k+l)!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)\,((\alpha\wedge\beta)\otimes\gamma)\,(X_{\pi(1)},\ldots,X_{\pi(k+l+m)})\\
&= \frac{1}{(k+l)!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)(\alpha\wedge\beta)(X_{\pi(1)},\ldots,X_{\pi(k+l)})\cdot\gamma(X_{\pi(k+l+1)},\ldots,X_{\pi(k+l+m)})\\
&= \frac{1}{(k+l)!k!l!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)\sum_{\pi'\in S_{k+l}}\mathrm{sgn}(\pi')(\alpha\otimes\beta)(X_{\pi'(\pi(1))},\ldots,X_{\pi'(\pi(k+l))})\cdot\\
&\qquad\gamma(X_{\pi(k+l+1)},\ldots,X_{\pi(k+l+m)})\,. \qquad \text{(A.15.7)}
\end{aligned}$$

We introduce a new permutation $\pi''\in S_{k+l+m}$ such that

$$\pi''(\pi(i))=\begin{cases}\pi'(\pi(i)) & \text{for}\quad 1\le i\le k+l\,,\\ \pi(i) & \text{for}\quad i>k+l\,,\end{cases}$$

which implies $\mathrm{sgn}(\pi'')=\mathrm{sgn}(\pi')$. One then obtains

$$\begin{aligned}
&((\alpha\wedge\beta)\wedge\gamma)(X_1,\ldots,X_{k+l+m})\\
&= \frac{1}{(k+l)!k!l!m!}\sum_{\pi'\in S_{k+l}}\mathrm{sgn}(\pi')\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)((\alpha\otimes\beta)\otimes\gamma)(X_{\pi''(\pi(1))},\ldots,X_{\pi''(\pi(k+l+m))})\,.
\end{aligned}$$

Set $\sigma:=\pi''\circ\pi$. Then $\mathrm{sgn}(\sigma)=\mathrm{sgn}(\pi'')\mathrm{sgn}(\pi)$, thus

$$\mathrm{sgn}(\pi)=\mathrm{sgn}(\sigma)\mathrm{sgn}(\pi'')=\mathrm{sgn}(\sigma)\mathrm{sgn}(\pi')$$

and we get

$$
\begin{aligned}
&((\alpha \wedge \beta) \wedge \gamma)(X_1, \dots, X_{k+l+m}) \\
&= \frac{1}{(k+l)!k!l!m!} \underbrace{\sum_{\pi' \in S_{k+l}} (\mathrm{sgn}(\pi'))^2}_{=(k+l)!} \\
&\quad \sum_{\sigma \in S_{k+l+m}} \mathrm{sgn}(\sigma)((\alpha \otimes \beta) \otimes \gamma)(X_{\sigma(1)}, \dots, X_{\sigma(k+l+m)}) \\
&= \frac{1}{k!l!m!} \sum_{\sigma \in S_{k+l+m}} \mathrm{sgn}(\sigma)((\alpha \otimes \beta) \otimes \gamma)(X_{\sigma(1)}, \dots, X_{\sigma(k+l+m)}) \, .
\end{aligned} \tag{A.15.8}
$$

A similar calculation gives

$$
\begin{aligned}
&(\alpha \wedge (\beta \wedge \gamma))(X_1, \dots, X_{k+l+m}) \\
&= \frac{1}{k!l!m!} \sum_{\sigma \in S_{k+l+m}} \mathrm{sgn}(\sigma)(\alpha \otimes (\beta \otimes \gamma))(X_{\sigma(1)}, \dots, X_{\sigma(k+l+m)}) \, ,
\end{aligned} \tag{A.15.9}
$$

and the associativity of the wedge product follows.

The above calculations lead to the following form of the wedge product of n forms, where associativity is hidden in the notation:

$$
\begin{aligned}
&(\alpha_1 \wedge \dots \wedge \alpha_n)(X_1, \dots, X_{k_1+\dots+k_n}) \\
&= \frac{1}{k_1! \cdots k_n!} \sum_{\pi \in S_{k_1+\dots+k_n}} \mathrm{sgn}(\pi)(\alpha_1 \otimes \dots \otimes \alpha_n)(X_{\pi(1)}, \dots, X_{\pi(k_1+\dots+k_n)}) \, ,
\end{aligned} \tag{A.15.10}
$$

where $\alpha_i \in \Omega^{k_i}(M)$ for $i = 1, \dots, n$ and $X_j \in \Gamma(TM)$ for $j = 1, \dots, k_1 + \dots + k_n$.

Let us apply the last formula to one-forms: if $\alpha_i \in \Omega^1(M)$ we have

$$
\begin{aligned}
(\alpha_1 \wedge \dots \wedge \alpha_n)(X_1, \dots, X_n) &= \sum_{\pi \in S_n} \mathrm{sgn}(\pi)(\alpha_1 \otimes \dots \otimes \alpha_n)(X_{\pi(1)}, \dots, X_{\pi(n)}) \\
&= \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^n \alpha_i(X_{\pi(i)}) \\
&= \det(\alpha_i(X_j)) \, ,
\end{aligned} \tag{A.15.11}
$$

where we have used the Leibniz formula for determinants. □

The *exterior derivative* of a differential form is defined as follows:

1. For a zero form f, the exterior derivative of f is its *usual differential* df.
2. For a k-form α, its *exterior derivative* $d\alpha$ is a $(k+1)$-form defined as

$$
d\alpha \equiv d\left(\frac{1}{k!}\alpha_{\mu_1\dots\mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}\right) := \frac{1}{k!} d\alpha_{\mu_1\dots\mu_k} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \, . \tag{A.15.12}
$$

Equivalently

$$
\begin{aligned}
d\alpha &= \frac{1}{k!}\partial_\beta \alpha_{\mu_1\dots\mu_k} dx^\beta \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \\
&= \frac{k+1}{(k+1)!}\partial_{[\beta}\alpha_{\mu_1\dots\mu_k]} dx^\beta \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \, ,
\end{aligned} \tag{A.15.13}
$$

which can also be written as

$$(d\alpha)_{\mu_1\dots\mu_{k+1}} = (k+1)\,\partial_{[\mu_1}\alpha_{\mu_2\dots\mu_{k+1}]}\,. \tag{A.15.14}$$

One easily checks, using $\partial_\alpha\partial_\beta y^\gamma = \partial_\beta\partial_\alpha y^\gamma$, that the exterior derivative behaves as a tensor under coordinate transformations. An "active way" of saying this is

$$d(\phi^*\alpha) = \phi^*(d\alpha)\,, \tag{A.15.15}$$

for any differentiable map ϕ. The tensorial character of d is also made clear by noting that for any torsion-free connection ∇ we have

$$\partial_{[\mu_1}\alpha_{\mu_2\dots\mu_{k+1}]} = \nabla_{[\mu_1}\alpha_{\mu_2\dots\mu_{k+1}]}\,. \tag{A.15.16}$$

Again by symmetry of second derivatives, it immediately follows from (A.15.12) that $d(df) = 0$ for any function, and subsequently also for any differential form:

$$d^2\alpha := d(d\alpha) = 0\,. \tag{A.15.17}$$

A coordinate-free definition of $d\alpha$ is

$$\begin{aligned} d\alpha(X_0, X_1, \dots, X_k) = &\sum_{0\le j\le k} (-1)^j X_j\left(\alpha(X_0,\dots,\widehat{X_j},\dots X_k)\right) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\widehat{X_j},\dots X_k)\,, \end{aligned} \tag{A.15.18}$$

where $\widehat{X_\ell}$ denotes the omission of the vector X_ℓ.

It is not too difficult to prove that if $\overset{k}{\alpha}$ is a k-form and $\overset{\ell}{\beta}$ is an ℓ-form, then the following version of the Leibniz rule holds:

$$d(\overset{k}{\alpha}\wedge\overset{\ell}{\beta}) = (d\,\overset{k}{\alpha})\wedge\overset{\ell}{\beta} + (-1)^k\,\overset{k}{\alpha}\wedge(d\,\overset{\ell}{\beta})\,. \tag{A.15.19}$$

In dimension n, let $\sigma \in \{\pm 1\}$ denote the parity of a permutation, set

$$\epsilon_{\mu_1\dots\mu_n} = \begin{cases} \sqrt{|\det g_{\alpha\beta}|}\,\sigma(\mu_1\dots\mu_n) & \text{if } (\mu_1\dots\mu_n) \text{ is a permutation of } (1\dots n); \\ 0 & \text{otherwise.} \end{cases}$$

The *Hodge dual* $\star\alpha$ of a k-form $\alpha = \alpha_{\mu_1\dots\mu_k}dx^{\mu_1}\otimes\dots\otimes dx^{\mu_k}$ is a $(n-k)$-form defined as

$$\star\alpha = \frac{1}{k!(n-k)!}\epsilon_{\mu_1\dots\mu_k\mu_{k+1}\dots\mu_n}\alpha^{\mu_1\dots\mu_k}dx^{\mu_{k+1}}\otimes\dots\otimes dx^{\mu_n}\,. \tag{A.15.20}$$

Equivalently,

$$\star\alpha_{\mu_{k+1}\dots\mu_n} = \frac{1}{k!(n-k)!}\epsilon_{\mu_1\dots\mu_k\mu_{k+1}\dots\mu_n}\alpha^{\mu_1\dots\mu_k}\,. \tag{A.15.21}$$

For example, in Euclidean three-dimensional space,

$$\star 1 = dx\wedge dy\wedge dz\,,\ \star dx = dy\wedge dz\,,\ \star(dy\wedge dz) = dx\,,\ \star(dx\wedge dy\wedge dz) = 1\,,$$

etc. In Minkowski four-dimensional spacetime we have, e.g.,

$$\star dt = -dx\wedge dy\wedge dz\,,\quad \star dx = -dy\wedge dz\wedge dt\,,$$
$$\star(dt\wedge dx) = -dy\wedge dz\,,\quad \star(dx\wedge dy) = -dz\wedge dt\,,\quad \star(dx\wedge dy\wedge dz) = -dt\,.$$

A.16 Submanifolds, integration, and Stokes' theorem

When integrating on manifolds, the starting observation is that the integral of a *scalar function* f with respect to the coordinate measure

$$d^n x := dx^1 \cdots dx^n$$

is *not* a coordinate-independent operation. This is due to the fact that, under a change of variables $x \mapsto \bar{x}(x)$, one has

$$\int_{\mathbb{R}^n} \bar{f}(\bar{x}) d^n \bar{x} = \int_{\mathbb{R}^n} \underbrace{\bar{f}(\bar{x}(x))}_{f(x)} |J_{x\mapsto\bar{x}}(x)| d^n x\,, \tag{A.16.1}$$

where the *Jacobian* $J_{x\mapsto\bar{x}}$ is the determinant of the Jacobi matrix,

$$J_{x\mapsto\bar{x}} = \left|\frac{\partial(\bar{x}^1,\ldots,\bar{x}^n)}{\partial(x^1,\ldots,x^n)}\right|\,.$$

Supposing that we have a metric

$$g = g_{ij}(x)dx^i dx^j = g_{ij}(x)\frac{\partial x^i}{\partial \bar{x}^k}(\bar{x}(x))\frac{\partial x^j}{\partial \bar{x}^\ell}(\bar{x}(x))\, d\bar{x}^k d\bar{x}^\ell = \bar{g}_{k\ell}(\bar{x}(x))d\bar{x}^k d\bar{x}^\ell \tag{A.16.2}$$

at our disposal, the problem can be cured by introducing the *metric measure*

$$d\mu_g := \sqrt{\det g_{ij}}\, d^n x\,. \tag{A.16.3}$$

Indeed, using

$$x(\bar{x}(x)) = x \implies \frac{\partial x^k}{\partial \bar{x}^\ell}(\bar{x}(x))\frac{\partial \bar{x}^\ell}{\partial x^i}(x) = \delta^k_i \implies J_{\bar{x}\mapsto x}(\bar{x}(x))\, J_{x\mapsto\bar{x}}(x) = 1\,,$$

it follows from (A.16.2) that

$$\sqrt{\det \bar{g}_{ij}(\bar{x}(x))} = \sqrt{\det g_{ij}(x)}|J_{\bar{x}\mapsto x}(\bar{x}(x))| = \frac{\sqrt{\det g_{ij}(x)}}{|J_{x\mapsto\bar{x}}(x)|}\,,$$

hence

$$d\mu_g \equiv \sqrt{\det g_{ij}(x)}\, d^n x = \sqrt{\det \bar{g}_{ij}(x(\bar{x}))}|J_{x\mapsto\bar{x}}(x)|\, d^n x\,. \tag{A.16.4}$$

This shows that

$$\int_{\mathbb{R}^n} f(x)\sqrt{\det g_{ij}}\, d^n x = \int_{\mathbb{R}^n} f(x)\sqrt{\det \bar{g}_{ij}}|J_{x\mapsto\bar{x}}(x)|\, d^n x\,.$$

Comparing with (A.16.1), this is equal to

$$\int_{\mathbb{R}^n} f(x)d\mu_g = \int_{\mathbb{R}^n} f(x(\bar{x}))\sqrt{\det \bar{g}_{ij}}\, d^n\bar{x} = \int_{\mathbb{R}^n} \bar{f}(\bar{x})d\mu_{\bar{g}}\,.$$

A similar formula holds for subsets of $\mathbb{R}^n$. We conclude that the metric measure $d\mu_g$ is the right thing to use when integrating scalars over a manifold.

Now, when defining conserved charges we have been integrating on submanifolds. The first naive thought would be to use the spacetime metric determinant as above for that, e.g., in spacetime dimension $n+1$,

$$\int_{\{x^0=0\}} f = \int_{\mathbb{R}^n} f(0, x^1, \ldots, x^n)\sqrt{\det g_{\mu\nu}}dx^1 \ldots dx^n \,.$$

This does not work because if we take g to be the Minkowski metric on $\mathbb{R}^n$, and replace x^0 by $\bar{x}^0$ using $x^0 = 2\bar{x}^0$, the only thing that will change in the last integral is the determinant $\sqrt{\det g_{\mu\nu}}$, giving a different value for the answer.

So, to proceed, it is useful to make first a short excursion into hypersurfaces, induced metrics and measures.

A.16.1 Hypersurfaces

A subset $\mathscr{S} \subset \mathscr{M}$ is called a *hypersurface* if near every point $p \in \mathscr{S}$ there exists a coordinate system $\{x^1, \ldots, x^n\}$ on a neighborhood $\mathscr{U}$ of p in $\mathscr{M}$ and a constant C such that

$$\mathscr{S} \cap \mathscr{U} = \{x^1 = C\} \,.$$

For example, any hyperplane $\{x^1 = \text{const}\}$ in $\mathbb{R}^n$ is a hypersurface. Similarly, a sphere $\{r = R\}$ in $\mathbb{R}^n$ is a hypersurface if $R > 0$.

Further examples include graphs,

$$x^1 = f(x^2, \ldots, x^{n-1}) \,,$$

which is seen by considering new coordinates $(\bar{x}^i) = (x^1 - f, x^2, \ldots x^n)$.

A standard result in analysis asserts that if φ is a differentiable function on an open set Ω such that $d\varphi$ nowhere zero on $\Omega \cap \{\varphi = c\}$ for some constant c, then

$$\Omega \cap \{\varphi = c\}$$

forms a hypersurface in Ω.

A vector $X \in T_p\mathscr{M}$, $p \in \mathscr{S}$, is said to be *tangent to* $\mathscr{S}$ if there exists a differentiable curve γ with image lying on $\mathscr{S}$, with $\gamma(0) = p$, such that $X = \dot{\gamma}(0)$. One denotes by $T\mathscr{S}$ the set of such vectors. Clearly, the bundle $T\mathscr{S}$ of all vectors tangent to $\mathscr{S}$, defined when $\mathscr{S}$ is viewed as a manifold on its own, is naturally diffeomorphic with the bundle $T\mathscr{S} \subset T\mathscr{M}$ just defined.

As an example, suppose that $\mathscr{S} = \{x^1 = C\}$ for some constant C, then $T\mathscr{S}$ is the collection of vectors defined along $\mathscr{S}$ for which $X^1 = 0$.

As another example, suppose that

$$\mathscr{S} = \{x^0 = f(x^i)\} \tag{A.16.5}$$

for some differentiable function f. Then a curve γ lies on $\mathscr{S}$ if and only if

$$\gamma^0 = f(\gamma^1, \ldots, \gamma^n) \,,$$

and so its tangent satisfies

$$\dot\gamma^0 = \partial_1 f \dot\gamma^1 + \ldots + \partial_n f \dot\gamma^n\,.$$

We conclude that X is tangent to $\mathscr{S}$ if and only if

$$X^0 = X^1\partial_1 f + \ldots + X^n \partial_n f = X^i\partial_i f \quad \Longleftrightarrow \quad X = X^i\partial_i f \partial_0 + X^i\partial_i\,. \tag{A.16.6}$$

Equivalently, the vectors

$$\partial_i f \partial_0 + \partial_i$$

form a basis of the tangent space $T\mathscr{S}$.

Finally, if

$$\mathscr{S} = \Omega \cap \{\varphi = c\} \tag{A.16.7}$$

then for any curve lying on $\mathscr{S}$ we have

$$\varphi(\gamma(s)) = c \quad \Longleftrightarrow \quad \dot\gamma^\mu \partial_\mu \varphi = 0 \;\text{ and }\; \varphi(\gamma(0)) = c\,.$$

Hence, a vector $X \in T_p\mathscr{M}$ is tangent to $\mathscr{S}$ if and only if $\varphi(p) = c$ and

$$X^\mu\partial_\mu\varphi = 0 \iff X(\varphi) = 0 \iff d\varphi(X) = 0\,. \tag{A.16.8}$$

A one-form α is said to *annihilate* $T\mathscr{S}$ if

$$\forall X \in T\mathscr{S} \qquad \alpha(X) = 0\,. \tag{A.16.9}$$

The set of such one-forms is called *the annihilator* of $T\mathscr{S}$, and denoted as $(T\mathscr{S})^o$. By elementary algebra, $(T\mathscr{S})^o$ is a one-dimensional subset of $T^*\mathscr{M}$. So, (A.16.8) can be rephrased as the statement that $d\varphi$ annihilates $T\mathscr{S}$.

A vector $Y \in T_p\mathscr{M}$ is said to be normal to $\mathscr{S}$ if Y is orthogonal to every vector in $X \in T_p\mathscr{S}$, where $T_p\mathscr{S}$ is viewed as a subset of $T_p\mathscr{M}$. Equivalently, the one form $g(Y,\cdot)$ annihilates $T_p\mathscr{S}$. If N has unit length, $g(N,N) \in \{-1,+1\}$, then N is said to be the unit normal. Thus,

$$\forall X \in T\mathscr{S} \qquad g(X,N) = 0\,, \quad g(N,N) = \epsilon \in \{\pm 1\}\,. \tag{A.16.10}$$

In Riemannian geometry only the plus sign is possible, and a unit normal vector always exists. This might not be the case in Lorentzian geometry: Indeed, consider the hypersurface

$$\mathscr{S} = \{t = x\} \subset \mathbb{R}^{1,1} \tag{A.16.11}$$

in two-dimensional Minkowski spacetime. A curve lying on $\mathscr{S}$ satisfies $\gamma^0(s) = \gamma^1(s)$, hence X is tangent to $\mathscr{S}$ if and only if $X^0 = X^1$. Let Y be orthogonal to $X \neq 0$, then

$$0 = \eta(X,Y) = X^0(-Y^0 + Y^1)\,,$$

whence

$$Y^0 = Y^1\,. \tag{A.16.12}$$

We conclude that, for non-zero X,

$$0 = \eta(X,Y) \quad \Rightarrow \quad Y \in T\mathscr{S},\ \text{in particular } 0 = \eta(Y,Y),$$

and so no such vector Y can have length one or minus one.

Since vectors of the form (A.16.12) are tangent to $\mathscr{S}$ as given by (A.16.11), we also reach the surprising conclusion that vectors normal to $\mathscr{S}$ coincide with vectors tangent to $\mathscr{S}$ in this case.

Suppose that the direction normal to $\mathscr{S}$ is timelike or spacelike. Then the *metric h induced by g on $\mathscr{S}$* is defined as

$$\forall\ X, Y \in T\mathscr{S} \quad h(X,Y) = g(X,Y)\,. \tag{A.16.13}$$

Hence, h coincides with g whenever both are defined, but we are only allowed to consider vectors tangent to $\mathscr{S}$ when using h.

Some comments are in order: If g is Riemannian, then normals to $\mathscr{S}$ are spacelike, and (A.16.13) defines a Riemannian metric on $\mathscr{S}$. For Lorentzian g's, it is easy to see that h is Riemannian if and only if vectors orthogonal to $\mathscr{S}$ are timelike, and then $\mathscr{S}$ is called *spacelike*. Similarly, h is Lorentzian if and only if vectors orthogonal to $\mathscr{S}$ are spacelike, and then $\mathscr{S}$ is called *timelike*. When the normal direction to $\mathscr{S}$ is null, then (A.16.13) defines a symmetric tensor on $\mathscr{S}$ with signature $(0,+,\cdots,+)$, which is degenerate and therefore not a metric; such hypersurfaces are called *null*, or *degenerate*.

If $\mathscr{S}$ is *not* degenerate, it comes equipped with a Riemannian or Lorentzian metric h. This metric defines a measure $d\mu_h$ which can be used to integrate over $\mathscr{S}$.

We are ready now to formulate the Stokes theorem for open bounded sets: Let Ω be a bounded open set with piecewise differentiable boundary and assume that there exists a well-defined field of exterior-pointing conormals $N = N_\mu dx^\mu$ to Ω. Then for any differentiable vector field X it holds that

$$\int_\Omega \nabla_\alpha X^\alpha d\mu_g = \int_{\partial\Omega} X^\mu N_\mu dS\,. \tag{A.16.14}$$

If $\partial\Omega$ is non-degenerate, N_μ can be normalised to have unit length, and then dS is the measure $d\mu_h$ associated with the metric h induced on $\partial\Omega$ by g.

The definition of dS for null hypersurfaces is somewhat more complicated. The key point is that (A.16.14) remains valid for a suitable measure dS on null components of the boundary. This measure is not uniquely defined by the geometry of the problem, but the product $N_\mu dS$ is.

Incidentally: In order to prove (A.16.14) on a smooth null hypersurface $\mathscr{N}$ one can proceed as follows. Let use denote by N any smooth field of null normals to $\mathscr{N}$; compare Appendix A.23, p. 316, where such a field is denoted by L. The field N is defined up to multiplication by a nowhere-vanishing smooth function. We can find an ON-frame $\{e_\mu\}$ so that the vector fields $e_2, \ldots e_n$ are tangent to $\mathscr{N}$ and orthogonal to N, with

$$N = e_0 + e_1\,. \tag{A.16.15}$$

Note that $\{e_0, e_1\}$ form an ON-basis of the space $\{e_2, \ldots, e_n\}^\perp$, and are thus defined up to changes of signs $(e_0, e_1) \mapsto (\pm e_0, \pm e_1)$ and two-dimensional Lorentz transformations. If $\mathscr{N} = \partial\Omega$ we choose e_0 to be outwards directed; then (A.16.15) determines the orientation of e_1.

Let $\{\theta^\mu\}$ be the dual basis, thus the volume form $d\mu_g$ is

$$d\mu_g = \theta^0 \wedge \cdots \wedge \theta^n\,.$$

Set

$$dS := -\theta^1 \wedge \cdots \wedge \theta^n|_{\mathscr{N}} \,,$$

where $(\cdots)|_{\mathscr{N}}$ denotes the pull-back to $\mathscr{N}$. It holds that

$$dS = -\theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n|_{\mathscr{N}} \,. \tag{A.16.16}$$

Indeed, we have $X \rfloor d\mu_g|_{\mathscr{N}} = 0$ for any vector field tangent to $\mathscr{N}$, in particular

$$0 = N \rfloor d\mu_g|_{\mathscr{N}} = \big(\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n - \theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n\big)\big|_{\mathscr{N}} \,,$$

which is (A.16.16).

In the formalism of differential forms Stokes' theorem on oriented manifolds reads

$$\int_\Omega \nabla_\mu X^\mu d\mu_g = \int_{\partial\Omega} X \rfloor d\mu_g \,. \tag{A.16.17}$$

If $\partial\Omega$ is null, in the adapted frame just described we have $X^\mu N_\mu = -X^0 + X^1$ and

$$\begin{aligned} X \rfloor d\mu_g|_{\partial\Omega} &= \big(X^0\theta^1 \wedge \cdots \wedge \theta^n - X^1\theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n\big)|_{\partial\Omega} = (-X^0 + X^1)dS \\ &= X^\mu N_\mu dS \,, \end{aligned} \tag{A.16.18}$$

as desired.

Since the left-hand side of (A.16.18) is independent of any choices made, so is the right-hand side. □

Remark A.16.2 The reader might wonder how (A.16.14) fits with the usual version of the divergence theorem

$$\int_\Omega \partial_\alpha X^\alpha d\mu_g = \int_{\partial\Omega} X^\mu dS_\mu \,, \tag{A.16.19}$$

which holds for sets Ω which can be covered by a single coordinate chart. For this we note the identity

$$\nabla_\mu X^\mu = \frac{1}{\sqrt{|\det g|}} \partial_\mu \big(\sqrt{|\det g|} X^\mu\big) \,, \tag{A.16.20}$$

which gives

$$\int_\Omega \nabla_\alpha X^\alpha d\mu_g = \int_\Omega \frac{1}{\sqrt{|\det g|}} \partial_\alpha \big(\sqrt{|\det g|} X^\alpha\big) \sqrt{|\det g|} d^n x = \int_\Omega \partial_\alpha \big(\sqrt{|\det g|} X^\alpha\big) d^n x \,. \tag{A.16.21}$$

This should make clear the relation between (A.16.19) and (A.16.14). □

A.17 Odd forms (densities)

The purpose of this section is to review the notion of an *odd* differential n-form on a manifold M; we follow the very clear approach of [270].

Locally, in a vicinity of a point x_0, an odd form may be defined as an equivalence class $[(\alpha_n, \mathcal{O})]$, where α_n is a differential n-form defined in a neighbourhood U and $\mathcal{O}$ is an orientation of U; the equivalence relation is given by:

$$(\alpha_n, \mathcal{O}) \sim (-\alpha_n, -\mathcal{O}) \,,$$

where $-\mathcal{O}$ denotes the orientation opposite to $\mathcal{O}$. Using a partition of unity, we may define odd forms globally, even if the manifold is non-orientable.

Odd differential n-forms on an m-dimensional manifold can be described using antisymmetric contravariant tensor densities of rank $r = (m - n)$ (see [254]). Indeed, if $f^{i_1 \dots i_r}$ are components of such a tensor density with respect to a coordinate system (x^i), then we may assign to f an odd n-form defined by the representative $(\alpha_n, \mathcal{O})$, where $\mathcal{O}$ is the local orientation carried by $(x^1, \dots, x^m)$ and

$$\alpha_n := f^{i_1 \dots i_r} \left(\frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_r}} \right) \lrcorner (\, dx^1 \wedge \dots \wedge \, dx^m) \, .$$

In particular, within this description scalar densities (*i.e.*, densities of rank $m - n = 0$) are odd forms of maximal rank, whereas vector densities are odd $(m - 1)$-forms.

Odd n-forms are designed to be integrated over *externally oriented* n-dimensional submanifolds. An exterior orientation of a submanifold Σ is an orientation of a bundle of tangent vectors transversal with respect to Σ. The integral of an odd form $\tilde{\alpha}_n = [(\alpha_n, \mathcal{O})]$ over a n-dimensional submanifold D with exterior orientation $\mathcal{O}_{\text{ext}}$ is defined as follows:

$$\int\limits_{(D,\mathcal{O}_{\text{ext}})} \tilde{\alpha}_n := \int\limits_{(D,\mathcal{O}_{\text{int}})} \alpha_n \ ,$$

where $\mathcal{O}_{\text{int}}$ is an internal orientation of D, such that $(\mathcal{O}_{\text{ext}}, \mathcal{O}_{\text{int}}) = \mathcal{O}$; it should be obvious that the result does not depend upon the choice of a representative. For example, a flow through a hypersurface depends usually upon its *exterior* orientation (given by a transversal vector) and does not feel any *interior* orientation. Similarly, the canonical formalism in field theory uses structures, which are defined in terms of flows through Cauchy hypersurfaces in spacetime. This is why canonical momenta are described by odd $(m-1)$-forms. The integrals of such forms are insensitive to any *internal* orientation of the hypersurfaces they are integrated upon, but are sensitive to a choice of the time arrow (*i.e.*, to its *exterior* orientation).

The Stokes theorem generalizes to odd forms in a straightforward way:

$$\int\limits_{(D,\mathcal{O}_{\text{ext}})} d\tilde{\alpha}_{n-1} = \int\limits_{\partial(D,\mathcal{O}_{\text{ext}})} \tilde{\alpha}_{n-1} \ ,$$

where $d[(\alpha_n, \mathcal{O})] := [(\, d\alpha_n, \mathcal{O})]$ and $\partial(D, \mathcal{O}_{\text{ext}})$ is the boundary of D, equipped with an exterior orientation inherited in the canonical way from $(D, \mathcal{O}_{\text{ext}})$. This means that if $(e_1, \dots, e_{m-n})$ is an oriented basis of vectors transversal to D and if f is a vector tangent to D, transversal to ∂D and pointing outwards of D, then the exterior orientation of $\partial(D, \mathcal{O}_{\text{ext}})$ is given by $(e_1, \dots, e_{m-n}, f)$.

A.18 Moving frames

A formalism which is very convenient for practical calculations is that of *moving frames*; it also plays a key role when considering spinors. By definition, a moving frame is a (locally defined) field of bases $\{e_a\}$ of TM such that the scalar products

$$g_{ab} := g(e_a, e_b) \tag{A.18.1}$$

are point independent. In most standard applications one assumes that the e_a's form an orthonormal basis, so that g_{ab} is a diagonal matrix with plus and minus ones on the diagonal. However, it is sometimes convenient to allow other such frames, *e.g.* with isotropic vectors being members of the frame.

It is customary to denote by $\omega^a{}_{bc}$ the associated connection coefficients:

$$\omega^a{}_{bc} := \theta^a(\nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_{bc} X^c e_a \,, \tag{A.18.2}$$

where, as elsewhere, $\{\theta^a(p)\}$ is a basis of T_p^*M dual to $\{e_a(p)\} \subset T_pM$; we will refer to θ^a as a *coframe*. The *connection one forms* $\omega^a{}_b$ are defined as

$$\omega^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_b(X) e_a \;;. \tag{A.18.3}$$

As always we use the metric to raise and lower indices, even though the $\omega^a{}_{bc}$'s do not form a tensor, so that

$$\omega_{abc} := g_{ad}\omega^e{}_{bc} \,, \quad \omega_{ab} := g_{ae}\omega^e{}_b \,. \tag{A.18.4}$$

When ∇ is metric compatible, the ω_{ab}'s are anti-antisymmetric: indeed, as the g_{ab}'s are point independent, for any vector field X we have

$$\begin{aligned}
0 = X(g_{ab}) = X(g(e_a, e_b)) &= g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) \\
&= g(\omega^c{}_a(X) e_c, e_b) + g(e_a, \omega^d{}_b(X) e_d) \\
&= g_{cb}\omega^c{}_a(X) + g_{ad}\omega^d{}_b(X) \\
&= \omega_{ba}(X) + \omega_{ab}(X) \,.
\end{aligned}$$

Hence

$$\boxed{\omega_{ab} = -\omega_{ba} \quad \Longleftrightarrow \quad \omega_{abc} = -\omega_{bac}} \,. \tag{A.18.5}$$

One can obtain a formula for the ω_{ab}'s in terms of Christoffels, the frame vectors and their derivatives: In order to see this, we note that

$$g(e_a, \nabla_{e_c} e_b) = g(e_a, \omega^d{}_{bc} e_d) = g_{ad}\omega^d{}_{bc} = \omega_{abc} \,. \tag{A.18.6}$$

Rewritten the other way round this gives an alternative equation for the ω's with all indices down:

$$\omega_{abc} = g(e_a, \nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \omega_{ab}(X) = g(e_a, \nabla_X e_b) \,. \tag{A.18.7}$$

Then, writing

$$e_a = e_a{}^\mu \partial_\mu \,,$$

we find

$$\begin{aligned}
\omega_{abc} &= g(e_a{}^\mu \partial_\mu, e_c{}^\lambda \nabla_\lambda e_b) \\
&= g_{\mu\sigma} e_a{}^\mu e_c{}^\lambda (\partial_\lambda e_b{}^\sigma + \Gamma^\sigma_{\lambda\nu} e_b{}^\nu) \,.
\end{aligned} \tag{A.18.8}$$

Next, it turns out that we can calculate the ω_{ab}'s in terms of the Lie brackets of the vector fields e_a, without having to calculate the Christoffel symbols. This shouldn't be too surprising, since an ON frame defines the metric uniquely. If ∇ has no torsion, from (A.18.7) we find

$$\omega_{abc} - \omega_{acb} = g(e_a, \nabla_{e_c} e_b - \nabla_{e_b} e_c) = g(e_a, [e_c, e_b])\,.$$

We can now carry out the usual cyclic-permutations calculation to obtain

$$\begin{aligned} \omega_{abc} - \omega_{acb} &= g(e_a, [e_c, e_b])\,, \\ -(\omega_{bca} - \omega_{bac}) &= -g(e_b, [e_a, e_c])\,, \\ -(\omega_{cab} - \omega_{cba}) &= -g(e_c, [e_b, e_a])\,. \end{aligned}$$

So, if the connection is the Levi-Civita connection, summing the three equations and using (A.18.5) leads to

$$\boxed{\omega_{cba} = \frac{1}{2}\Big(g(e_a, [e_c, e_b]) - g(e_b, [e_a, e_c]) - g(e_c, [e_b, e_a])\Big)}\,. \qquad \text{(A.18.9)}$$

Equations (A.18.8)-(A.18.9) provide explicit expressions for the ω's; yet another formula can be found in (A.18.11) below. While it is useful to know that there are such expressions, and while those expressions are useful to estimate things for PDE purposes, they are rarely used for practical calculations; see Example A.18.3 for more comments about that last issue.

It turns out that one can obtain a simple expression for the torsion of ω using exterior differentiation. Recall that if α is a one-form, then its exterior derivative $d\alpha$ can be calculated using the formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])\,. \qquad \text{(A.18.10)}$$

Exercice A.18.1 Use (A.18.9) and (A.18.10) to show that

$$\omega_{cba} = \frac{1}{2}\Big(- \eta_{ad}\, d\theta^d(e_c, e_b) + \eta_{bd}\, d\theta^d(e_a, e_c) + \eta_{cd}\, d\theta^d(e_b, e_a)\Big)\,. \qquad \text{(A.18.11)}$$

□

We set

$$T^a(X,Y) := \theta^a(T(X,Y))\,,$$

and using (A.18.10) together with the definition (A.9.16) of the torsion tensor T we calculate as follows:

$$\begin{aligned} T^a(X,Y) &= \theta^a(\nabla_X Y - \nabla_Y X - [X,Y]) \\ &= X(Y^a) + \omega^a{}_b(X)Y^b - Y(X^a) - \omega^a{}_b(Y)X^b - \theta^a([X,Y]) \\ &= X(\theta^a(Y)) - Y(\theta^a(X)) - \theta^a([X,Y]) + \omega^a{}_b(X)\theta^b(Y) - \omega^a{}_b(Y)\theta^b(X) \\ &= d\theta^a(X,Y) + (\omega^a{}_b \wedge \theta^b)(X,Y)\,. \end{aligned}$$

It follows that

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b\,. \qquad \text{(A.18.12)}$$

In particular when the torsion vanishes we obtain the so-called *Cartan's first structure equation*

$$\boxed{d\theta^a + \omega^a{}_b \wedge \theta^b = 0}\,. \qquad \text{(A.18.13)}$$

EXAMPLE A.18.2 As a simple example, we consider a two-dimensional metric of the form
$$g = dx^2 + e^{2f} dy^2 \,, \tag{A.18.14}$$
where f could possibly depend upon x and y. A natural frame is given by
$$\theta^1 = dx \,, \quad \theta^2 = e^f dy \,.$$
The first Cartan structure equations read
$$0 = \underbrace{d\theta^1}_{0} + \omega^1{}_b \wedge \theta^b = \omega^1{}_2 \wedge \theta^2 \,,$$
since $\omega^1{}_1 = \omega_{11} = 0$ by antisymmetry, and
$$0 = \underbrace{d\theta^2}_{e^f \partial_x f dx \wedge dy} + \omega^2{}_b \wedge \theta^b = \partial_x f \theta^1 \wedge \theta^2 + \omega^2{}_1 \wedge \theta^1 \,.$$
It should then be clear that both equations can be solved by choosing ω_{12} proportional to θ^2, and such an ansatz leads to
$$\omega_{12} = -\omega_{21} = -\partial_x f \, \theta^2 = -\partial_x (e^f) \, dy \,. \tag{A.18.15}$$
We continue this example on p. 288. □

EXAMPLE A.18.3 As another example of the moving frame technique we consider (the most general) three-dimensional spherically symmetric metric
$$g = e^{2\beta(r)} dr^2 + e^{2\gamma(r)} d\theta^2 + e^{2\gamma(r)} \sin^2 \theta d\varphi^2 \,. \tag{A.18.16}$$
There is an obvious choice of ON coframe for g given by
$$\theta^1 = e^{\beta(r)} dr \,, \ \theta^2 = e^{\gamma(r)} d\theta \,, \ \theta^3 = e^{\gamma(r)} \sin\theta d\varphi \,, \tag{A.18.17}$$
leading to
$$g = \theta^1 \otimes \ \theta^1 + \theta^2 \otimes \ \theta^2 + \theta^3 \otimes \ \theta^3 \,,$$
so that the frame e_a dual to the θ^a's will be ON, as desired:
$$g_{ab} = g(e_a, e_b) = \mathrm{diag}(1, 1, 1) \,.$$
The idea of the calculation which we are about to do is the following: there is only one connection which is compatible with the metric, and which is torsion free. If we find a set of one forms ω_{ab} which exhibit the properties just mentioned, then they have to be the connection forms of the Levi-Civita connection. As shown in the calculation leading to (A.18.5), the compatibility with the metric will be ensured if we require
$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \,,$$
$$\omega_{12} = -\omega_{21} \,, \quad \omega_{13} = -\omega_{31} \,, \quad \omega_{23} = -\omega_{32} \,.$$
Next, we have the equations for the vanishing of torsion:
$$\begin{aligned} 0 = d\theta^1 &= -\underbrace{\omega^1{}_1}_{=0} \theta^1 - \omega^1{}_2 \theta^2 - \omega^1{}_3 \theta^3 \\ &= -\omega^1{}_2 \theta^2 - \omega^1{}_3 \theta^3 \,, \\ d\theta^2 &= \gamma' e^\gamma dr \wedge d\theta = \gamma' e^{-\beta} \theta^1 \wedge \theta^2 \end{aligned}$$

$$
\begin{aligned}
&= -\underbrace{\omega^2{}_1}_{=-\omega^1{}_2}\theta^1 - \underbrace{\omega^2{}_2}_{=0}\theta^2 - \omega^2{}_3\theta^3 \\
&= \omega^1{}_2\theta^1 - \omega^2{}_3\theta^3\,, \\
d\theta^3 &= \gamma' e^{\gamma}\sin\theta\, dr\wedge d\varphi + e^{\gamma}\cos\theta\, d\theta\wedge d\varphi = \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3 \\
&= -\underbrace{\omega^3{}_1}_{=-\omega^1{}_3}\theta^1 - \underbrace{\omega^3{}_2}_{=-\omega^2{}_3}\theta^2 - \underbrace{\omega^3{}_3}_{=0}\theta^3 \\
&= \omega^1{}_3\theta^1 + \omega^2{}_3\theta^2\,.
\end{aligned}
$$

Summarising,

$$
\begin{aligned}
-\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \qquad\qquad &= 0\,, \\
\omega^1{}_2\theta^1 \qquad\qquad - \omega^2{}_3\theta^3 &= \gamma' e^{-\beta}\theta^1\wedge\theta^2\,, \\
\omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 &= \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3\,.
\end{aligned}
$$

It should be clear from the first and second line that an $\omega^1{}_2$ proportional to θ^2 should do the job; similarly from the first and third line one sees that an $\omega^1{}_3$ proportional to θ^3 should work. It is then easy to find the relevant coefficient, as well as to find $\omega^2{}_3$:

$$
\begin{aligned}
\omega^1{}_2 &= -\gamma' e^{-\beta}\theta^2 = -\gamma' e^{-\beta+\gamma}d\theta\,, && (A.18.18a) \\
\omega^1{}_3 &= -\gamma' e^{-\beta}\theta^3 = -\gamma' e^{-\beta+\gamma}\sin\theta\, d\varphi\,, && (A.18.18b) \\
\omega^2{}_3 &= -e^{-\gamma}\cot\theta\,\theta^3 = -\cos\theta\, d\varphi\,. && (A.18.18c)
\end{aligned}
$$

We continue this example on p. 288. □

It is convenient to define *curvature two-forms*:

$$\Omega^a{}_b = R^a{}_{bcd}\theta^c\otimes\theta^d = \frac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d\,. \qquad (A.18.19)$$

The *second Cartan structure equation* reads

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b}\,. \qquad (A.18.20)$$

This identity is easily verified using (A.18.10):

$$
\begin{aligned}
\Omega^a{}_b(X,Y) &= \frac{1}{2}R^a{}_{bcd}\underbrace{\theta^c\wedge\theta^d(X,Y)}_{=X^cY^d-X^dY^c} \\
&= R^a{}_{bcd}X^cY^d \\
&= \theta^a(\nabla_X\nabla_Y e_b - \nabla_Y\nabla_X e_b - \nabla_{[X,Y]}e_b) \\
&= \theta^a(\nabla_X(\omega^c{}_b(Y)e_c) - \nabla_Y(\omega^c{}_b(X)e_c) - \omega^c{}_b([X,Y])e_c) \\
&= \theta^a\Big(X(\omega^c{}_b(Y))e_c + \omega^c{}_b(Y)\nabla_X e_c \\
&\qquad -Y(\omega^c{}_b(X))e_c - \omega^c{}_b(X)\nabla_Y e_c - \omega^c{}_b([X,Y])e_c\Big) \\
&= X(\omega^a{}_b(Y)) + \omega^c{}_b(Y)\omega^a{}_c(X) \\
&\qquad -Y(\omega^a{}_b(X)) - \omega^c{}_b(X)\omega^a{}_c(Y) - \omega^a{}_b([X,Y]) \\
&= \underbrace{X(\omega^a{}_b(Y)) - Y(\omega^a{}_b(X)) - \omega^a{}_b([X,Y])}_{=d\omega^a{}_b(X,Y)} \\
&\qquad +\omega^a{}_c(X)\omega^c{}_b(Y) - \omega^a{}_c(Y)\omega^c{}_b(X) \\
&= (d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b)(X,Y)\,.
\end{aligned}
$$

Equation (A.18.20) provides an efficient way of calculating the curvature tensor of any metric.

EXAMPLE A.18.4 In dimension two the only non-vanishing components of $\omega^a{}_b$ are $\omega^1{}_2 = -\omega^2{}_1$, and it follows from (A.18.20) that

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^1{}_a \wedge \omega^a{}_2 = d\omega^1{}_2 \,. \tag{A.18.21}$$

In particular (assuming that θ^2 is dual to a spacelike vector, whatever the signature of the metric)

$$\begin{aligned} R\, d\mu_g &= R\,\theta^1 \wedge \theta^2 = 2R^{12}{}_{12}\theta^1 \wedge \theta^2 = R^1{}_{2ab}\theta^a \wedge \theta^b = 2\Omega^1{}_2 \\ &= 2d\omega^1{}_2 \,, \end{aligned} \tag{A.18.22}$$

where $d\mu_g$ is the volume two-form. □

INCIDENTALLY: EXAMPLE A.18.2 CONTINUED We have seen that the connection one-forms for the metric

$$g = dx^2 + e^{2f} dy^2 \tag{A.18.23}$$

read

$$\omega_{12} = -\omega_{21} = -\partial_x f\, \theta^2 = -\partial_x(e^f)\, dy \,.$$

By symmetry the only non-vanishing curvature two-forms are $\Omega_{12} = -\Omega_{21}$. From (A.18.20) we find

$$\Omega_{12} = d\omega_{12} + \underbrace{\omega_{1b} \wedge \omega^b{}_2}_{=\omega_{12}\wedge\omega^2{}_2=0} = -\partial_x^2(e^f)\, dx \wedge dy = -e^{-f}\partial_x^2(e^f)\, \theta^1 \wedge \theta^2 \,.$$

We conclude that

$$R_{1212} = -e^{-f}\partial_x^2(e^f) \,. \tag{A.18.24}$$

(Compare Example A.12.6, p. 261.) For instance, if g is the unit round metric on the two-sphere, then $e^f = \sin x$, and $R_{1212} = 1$. If $e^f = \sinh x$, then g is the canonical metric on hyperbolic space, and $R_{1212} = -1$. Finally, the function $e^f = \cosh x$ defines a *hyperbolic wormhole*, with again $R_{1212} = -1$. □

INCIDENTALLY: EXAMPLE A.18.3 CONTINUED: From (A.18.18) we find:

$$\begin{aligned} \Omega^1{}_2 &= d\omega^1{}_2 + \underbrace{\omega^1{}_1}_{=0} \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \underbrace{\omega^2{}_2}_{=0} + \underbrace{\omega^1{}_3 \wedge \omega^3{}_2}_{\sim\theta^3\wedge\theta^3=0} \\ &= -d(\gamma' e^{-\beta+\gamma} d\theta) \\ &= -(\gamma' e^{-\beta+\gamma})' dr \wedge d\theta \\ &= -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma}\theta^1 \wedge \theta^2 \\ &= \sum_{a<b} R^1{}_{2ab}\theta^a \wedge \theta^b \,, \end{aligned}$$

which shows that the only non-trivial coefficient (up to permutations) with the pair 12 in the first two slots is

$$R^1{}_{212} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \,. \tag{A.18.25}$$

A similar calculation, or arguing by symmetry, leads to

$$R^1{}_{313} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \,. \tag{A.18.26}$$

Finally,

$$\begin{aligned}\Omega^2{}_3 &= d\omega^2{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 + \underbrace{\omega^2{}_2}_{=0} \wedge \omega^2{}_3 + \omega^2{}_3 \wedge \underbrace{\omega^3{}_3}_{=0} \\ &= -d(\cos\theta\, d\varphi) + (\gamma' e^{-\beta}\theta^2) \wedge (-\gamma' e^{-\beta}\theta^3) \\ &= (e^{-2\gamma} - (\gamma')^2 e^{-2\beta})\theta^2 \wedge \theta^3\,,\end{aligned}$$

yielding

$$R^2{}_{323} = e^{-2\gamma} - (\gamma')^2 e^{-2\beta}\,. \tag{A.18.27}$$

The curvature scalar can easily be calculated now to be

$$\begin{aligned}R = R^{ij}{}_{ij} &= 2(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}) \\ &= -4(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} + 2(e^{-2\gamma} - (\gamma')^2 e^{-2\beta})\,. \end{aligned} \tag{A.18.28}$$

□

Example A.18.7 Consider an n-dimensional Riemannian metric of the form

$$g = e^{2h(r)} dr^2 + e^{2f(r)} \underbrace{\mathring{h}_{AB}(x^C) dx^A dx^B}_{=:\mathring{h}}\,. \tag{A.18.29}$$

Let $\mathring{\theta}^A$ be an ON-frame for $\mathring{h}$, with corresponding connection coefficients $\mathring{\omega}^A{}_B$:

$$d\mathring{\theta}^A + \mathring{\omega}^A{}_B \wedge \mathring{\theta}^B = 0\,.$$

Set

$$\theta^1 = e^h dr\,, \quad \theta^A = e^f \mathring{\theta}^A\,.$$

Then the first structure equations,

$$\underbrace{d\theta^1}_{0} + \omega^1{}_B \wedge \theta^B = 0\,,$$

$$d(e^f \mathring{\theta}^A) + e^h \omega^A{}_1 \wedge dr + e^f \omega^A{}_B \wedge \mathring{\theta}^B = 0\,,$$

are solved by

$$\omega^A{}_1 = e^{-h}(e^f)'\, \mathring{\theta}^A\,, \quad \omega^A{}_B = \mathring{\omega}^A{}_B\,. \tag{A.18.30}$$

This leads to

$$\begin{aligned}\Omega^A{}_1 &= d\omega^A{}_1 + \omega^A{}_B \wedge \omega^B{}_1 \\ &= e^{-h-f}(e^{-h}(e^f)')'\, \theta^1 \wedge \theta^A\,, &\text{(A.18.31)} \\ \Omega^A{}_B &= \mathring{\Omega}^A{}_B - e^{-2h-2f}((e^f)')^2\, \theta^A \wedge \theta_B\,, &\text{(A.18.32)}\end{aligned}$$

where $\mathring{\Omega}^A{}_B$ are the curvature two-forms of the metric $\mathring{h}$,

$$\mathring{\Omega}^A{}_B = \frac{1}{2}\mathring{R}^A{}_{BCD}\, \mathring{\theta}^A \wedge \mathring{\theta}^B = \frac{e^{-2f}}{2}\mathring{R}^A{}_{BCD}\, \theta^A \wedge \theta^B\,. \tag{A.18.33}$$

Hence

$$\begin{aligned}R^A{}_{1B1} &= -e^{-h-f}(e^{-h}(e^f)')'\, \delta^A_B\,, &\text{(A.18.34)} \\ R^A{}_{1BC} &= 0 \;=\; R^1{}_B\,, &\text{(A.18.35)} \\ R^A{}_{BCD} &= e^{-2f}\mathring{R}^A{}_{BCD} - e^{-2h-2f}((e^f)')^2 \delta^A_{[C} g_{D]B}\,, &\text{(A.18.36)} \\ R^A{}_C &= -e^{-h-f}\big((e^{-h}(e^f)')' + (n-2)e^{-h-f}((e^f)')^2\big)\delta^A_C \\ &\quad + e^{-2f}\mathring{R}^A{}_C\,, &\text{(A.18.37)} \\ R^1{}_1 &= -(n-1)e^{-h-f}(e^{-h}(e^f)')'\,, &\text{(A.18.38)} \\ R &= -(n-1)e^{-h-f}\big(2(e^{-h}(e^f)')' + (n-2)e^{-h-f}((e^f)')^2\big) \\ &\quad + e^{-2f}\mathring{R}\,. &\text{(A.18.39)}\end{aligned}$$

Let g be the space-part of the Birmingham metrics (4.6.1)-(4.6.2), p. 150, thus g takes the form (A.18.29) with

$$e^f = r\,, \quad e^{-2h} = \beta - \frac{2m}{r^{n-2}} - \epsilon \frac{r^2}{\ell^2}\,, \qquad \epsilon \in \{0, \pm 1\}\,, \tag{A.18.40}$$

where β, m and ℓ are real constants. Then

$$R^A{}_{1B1} = \big(\frac{\epsilon}{\ell^2} - m(n-2)r^{-n}\big)\delta^A_B\,, \tag{A.18.41}$$

$$R^A{}_{1BC} = 0 = R^1{}_B\,, \tag{A.18.42}$$

$$R^A{}_{BCD} = \mathring{R}^A{}_{BCD} - \big(\frac{\beta}{r^2} - \frac{\epsilon}{\ell^2} + 2mr^{-n}\big)\delta^A_{[C}g_{D]B}\,, \tag{A.18.43}$$

$$R^A{}_B = \frac{\ell^2(n-2)\left(mr^2 - \beta r^n\right) + \epsilon(n-1)r^{n+2}}{\ell^2 r^{n+2}}\delta^A_B + r^{-2}\mathring{R}^A{}_B\,, \tag{A.18.44}$$

$$R^1{}_1 = (n-1)\big(\frac{\epsilon}{\ell^2} - m(n-2)r^{-n}\big)\,, \tag{A.18.45}$$

$$R = \frac{(n-1)\left(\epsilon n r^2 - \beta\ell^2(n-2)\right)}{\ell^2 r^2} + r^{-2}\mathring{R}\,. \tag{A.18.46}$$

If $\mathring{h}$ is Einstein, with

$$\mathring{R}_{AB} = (n-2)\beta\mathring{h}_{AB}\,, \tag{A.18.47}$$

the last formulae above simplify to

$$R^A{}_B = \frac{\ell^2(n-2)r^{-n-2}\left(\beta(n-2)r^n + mr^2\right) + \epsilon(n-1)}{\ell^2}\delta^A_B\,, \tag{A.18.48}$$

$$R = \frac{\epsilon(n-1)n}{\ell^2}\,. \tag{A.18.49}$$

□

Example A.18.8 We can use (A.18.11),

$$\omega_{cba} = \frac{1}{2}\Big(- \eta_{ad}\, d\theta^d(e_c, e_b) + \eta_{bd}\, d\theta^d(e_a, e_c) + \eta_{cd}\, d\theta^d(e_b, e_a)\Big)\,, \tag{A.18.50}$$

to determine how the curvature tensor transforms under conformal rescalings. For this let $g = \eta_{ab}\theta^a\theta^b$ with $d\eta_{ab} = 0$, and let

$$\overline{g} = e^{2f} g = \eta_{ab}\underbrace{e^f\theta^a}_{=:\overline{\theta}a}\otimes e^f\theta^b \equiv \eta_{ab}\overline{\theta}^a\overline{\theta}^b\,. \tag{A.18.51}$$

If the vector fields $\{e_a\}$ form a basis dual to the basis $\{\theta^a\}$, then the vector fields $\overline{e}_a = e^{-f}e_a$ provide a basis dual to $\{\overline{\theta}^b\}$,

$$\begin{aligned}\overline{\omega}_{cba} &= \frac{1}{2}\Big(- \eta_{ad}\, d(e^f\theta^d)(e^{-f}e_c, e^{-f}e_b) + \eta_{bd}\, d(e^f\theta^d)(e^{-f}e_a, e^{-f}e_c) \\ &\quad + \eta_{cd}\, d(e^f\theta^d)(e^{-f}e_b, e^{-f}e_a)\Big) \\ &= e^{-f}\left(\omega_{cba} + \frac{1}{2}\Big(- \eta_{ad}\,(df\wedge\theta^d)(e_c, e_b) + \eta_{bd}\,(df\wedge\theta^d)(e_a, e_c) + \eta_{cd}\,(df\wedge\theta^d)(e_b, e_a)\Big)\right) \\ &= e^{-f}\left(\omega_{cba} - \eta_{a[b}e_{c]}(f) + \eta_{b[c}e_{a]}(f) + \eta_{c[a}e_{b]}(f)\right) \\ &= e^{-f}\left(\omega_{cba} - \eta_{ab}e_c(f) + \eta_{ac}e_b(f)\right)\,.\end{aligned} \tag{A.18.52}$$

Equivalently,

$$\overline{\omega}_{cb} = \overline{\omega}_{cba}\overline{\theta}^a = e^f\overline{\omega}_{cba}\theta^a = \omega_{cb} + \big(\eta_{ac}\nabla_b f - \eta_{ab}\nabla_c f\big)\theta^a\,. \tag{A.18.53}$$

Taking the exterior derivative one finds

$$\begin{aligned}\overline{\Omega}_{cb} &= \Omega_{cb} + \Big(\eta_{ab}\nabla_d\nabla_c f - \eta_{ac}\nabla_d\nabla_b f \\ &\quad +\eta_{ac}\nabla_d f\nabla_b f + \eta_{db}\nabla_c f\nabla_a f - \eta_{ac}\eta_{db}|df|^2_g\Big)\theta^a\wedge\theta^d\,.\end{aligned} \tag{A.18.54}$$

Reexpressed in terms of the Riemann tensor, this reads

$$\begin{aligned}e^{2f}\overline{R}_{cbad} &= R_{cbad} + 2\Big(\eta_{b[a}\nabla_{d]}\nabla_c f - \eta_{c[a}\nabla_{d]}\nabla_b f \\ &\quad +\eta_{c[a}\nabla_{d]} f\nabla_b f + \eta_{b[d}\nabla_{a]} f\nabla_c f - \eta_{c[a}\eta_{d]b}|df|^2_g\Big)\,,\end{aligned} \tag{A.18.55}$$

where the components $\overline{R}_{cbad}$ are taken with respect to a $\overline{g}$-ON frame, and all components of the right-hand side are taken with respect to a g-ON frame. Taking traces we obtain, in dimension d,

$$e^{2f}\overline{R}_{ac} = R_{ac} + (2-d)\Big(\nabla_a\nabla_c f - \nabla_a f\nabla_c f + |df|^2_g\eta_{ac}\Big) - \Delta_g f\eta_{ac}\,, \tag{A.18.56}$$

$$e^{2f}\overline{R} = R + (1-d)\big(2\Delta_g f + (d-2)|df|^2_g\big)\,. \tag{A.18.57}$$

□

The Bianchi identities have a particularly simple proof in the moving frame formalism. For this, let ψ^a be any vector-valued differential form, and define

$$D\psi^a = d\psi^a + \omega^a{}_b\wedge\psi^b\,. \tag{A.18.58}$$

Thus, in this notation the vanishing of torsion reads

$$D\theta^a = 0\,. \tag{A.18.59}$$

Whether or not the torsion vanishes, we find

$$\begin{aligned}D\tau^a &= d\tau^a + \omega^a{}_b\wedge\tau^b = d(d\theta^a + \omega^a{}_b\wedge\theta^b) + \omega^a{}_c\wedge(d\theta^c + \omega^c{}_b\wedge\theta^b) \\ &= d\omega^a{}_b\wedge\theta^b - \omega^a{}_b\wedge d\theta^b + \omega^a{}_c\wedge(d\theta^c + \omega^c{}_b\wedge\theta^b) \\ &= \Omega^a{}_b\wedge\theta^b\,.\end{aligned}$$

If the torsion vanishes the left-hand side is zero, and we find

$$\Omega^a{}_b\wedge\theta^b = 0\,. \tag{A.18.60}$$

This is equivalent to the first Bianchi identity:

$$0 = \Omega^a{}_b\wedge\theta^b = \frac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d\wedge\theta^b = R^a{}_{[bcd]}\theta^c\wedge\theta^d\wedge\theta^b \quad\Longleftrightarrow\quad R^a{}_{[bcd]} = 0\,. \tag{A.18.61}$$

Next, for any differential form α_b with two-frame indices, such as the curvature two-form, we define

$$D\alpha^a{}_b := d\alpha^a{}_b + \omega^a{}_c\wedge\alpha^c{}_b - \omega^c{}_b\wedge\alpha^a{}_c\,. \tag{A.18.62}$$

(The reader will easily work-out the obvious generalisation of this definition to differential forms with any number of frame indices.) For the curvature two-form we find

$$\begin{aligned} D\Omega^a{}_b &= d(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\ &= d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\ &= (\Omega^a{}_c - \omega^a{}_e \wedge \omega^e{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_e \wedge \omega^e{}_b) \\ &\quad + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c = 0\,. \end{aligned}$$

Thus

$$D\Omega^a{}_b = 0\,, \tag{A.18.63}$$

Let us show that this is equivalent to the second Bianchi identity:

$$\begin{aligned} 0 \ &= D\Omega^a{}_b = \frac{1}{2}R^a{}_{bc;d}\theta^d \wedge \theta^b \wedge \theta^c = \frac{1}{2}R^a{}_{[bc;d]}\theta^d \wedge \theta^b \wedge \theta^c \\ \iff &\quad R^a{}_{[bc;d]} = 0\,. \end{aligned} \tag{A.18.64}$$

Here the only not-obviously-apparent fact is, if any, the second equality in the first line of(A.18.64):

$$\begin{aligned} D\Omega^a{}_b &= \frac{1}{2}\Big(d(R^a{}_{bef}\theta^e \wedge \theta^f) + \omega^a{}_c \wedge R^c{}_{bef}\,\theta^e \wedge \theta^f - \omega^c{}_b \wedge R^a{}_{cef}\,\theta^e \wedge \theta^f\Big) \\ &= \frac{1}{2}\Big(\underbrace{dR^a{}_{bef}}_{e_k(R^a{}_{bef})\,\theta^k} \wedge\theta^e \wedge \theta^f + R^a{}_{bef}\underbrace{d\theta^e}_{-\omega^e{}_k\wedge\theta^k} \wedge\theta^f + R^a{}_{bef}\theta^e \wedge \underbrace{d\theta^f}_{-\omega^f{}_k\wedge\theta^k} \\ &\quad + R^c{}_{bef}\,\omega^a{}_c \wedge \theta^e \wedge \theta^f - R^a{}_{cef}\,\omega^c{}_b \wedge \theta^e \wedge \theta^f\Big) \\ &= \frac{1}{2}\nabla_{e_k}R^a{}_{bef}\,\theta^k \wedge \theta^e \wedge \theta^f\,, \end{aligned} \tag{A.18.65}$$

as desired.

A.19 Lovelock Theorems

In [191] Lovelock showed that the equations

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi T_{ij}$$

are the only second-order equations for the metric in spacetime dimension four in which the "matter conservation law" $\nabla_i T^i{}_j = 0$ is a consequence of the equations.[3] In this Appendix we will present Lovelock's results, and derive some of them.

[3] Compare Remark A.19.2, Proposition A.19.4 and Theorem A.19.5 below.

A.19.1 Lovelock Lagrangeans

As pointed out by Zumino [282], the moving-frame formalism of Appendix A.18 is particularly efficient in proving the "if part" of the following theorem of Lovelock, in which Lagrangeans in spacetime dimension d are considered to be d-forms:

THEOREM A.19.1 (Lovelock [190]) *Let the spacetime dimension be d. A diffeomorphism-invariant Lagrangean $\mathscr{L}$ depending only upon the metric and its derivatives up to order two leads to second-order field equations for the metric if and only if $\mathscr{L}$ is a linear combination of the volume form and of the following d-forms, with $2k+2 \le d$:*

$$\mathscr{L}_k = \epsilon^{a_1\ldots a_d}\Omega_{a_1a_2} \wedge \cdots \wedge \Omega_{a_{2k+1}a_{2k+2}} \wedge \theta_{a_{2k+3}} \wedge \cdots \wedge \theta_{a_d} \,. \tag{A.19.1}$$

REMARK A.19.2 A pure volume-form part of a Lagrangean contributes a cosmological constant to the field equations, while the variation of $\mathscr{L}_0$ produces the Einstein tensor; cf. (A.19.6) and (A.19.10) below.

We emphasise that in each dimension there is only a finite number of such Lagrangeans, e.g. in $d = 4$ only $k = 0$ and $k = 1$ occur. Proposition A.19.4 below shows that the case $k = 1$ and $d = 4$ is irrelevant as far as the field equations are concerned, as it does not contribute to those equations.

INCIDENTALLY: Curiously enough, the integrand of the *Weyl tube-volume formula* [277] involves only linear combinations of the Lovelock d-forms $\mathscr{L}_k$. This raises the perplexing question of existence of a relation between the formula and the Lovelock theorems.

PROOF: We will only prove the easier part of the theorem, namely that the variation of $\mathscr{L}_k$ produces a tensor which depends at most upon two derivatives of the metric.

A variation of the metric will produce a variation $\delta\theta^a$ of the moving frame and an associated variation $\delta\omega^a{}_b$ of the connection coefficients. From the form of $\mathscr{L}_k$ we see that the derivatives $\partial_a\delta\theta^a$ will enter in the variation $\delta\mathscr{L}_k$ through the variations $\delta\omega^a{}_b$ only. The contribution of undifferentiated variations $\delta\theta^a$ will only produce terms which contain at most two derivatives of the metric. Therefore, to establish the claim it suffices to show that the contribution of the variations $\delta\omega^a{}_b$ of the connection coefficients to the variation of the action vanishes.

From (A.18.20) we find

$$\delta\Omega^a{}_b = d(\delta\omega^a{}_b) + \delta\omega^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \delta\omega^c{}_b = D\delta\omega^a{}_b \,. \tag{A.19.2}$$

Using the vanishing of the torsion, $D\theta^a = 0$, and the second Bianchi identity, $D\Omega^a{}_b = 0$, one finds that the variation $\delta\mathscr{L}_k$ of $\mathscr{L}_k$ associated with the $\delta\omega^a{}_b$'s is a full divergence and therefore will not contribute to the field equations:

$$\delta\mathscr{L}_k = \epsilon^{a_1\ldots a_d}\Big(\delta\Omega_{a_1a_2} \wedge \cdots \wedge \Omega_{a_{2k+1}a_{2k+2}} \wedge \theta_{a_{2k+3}} \wedge \cdots \wedge \theta_{a_d} + \cdots$$

$$
\begin{aligned}
&+\Omega_{a_1a_2}\wedge\cdots\wedge\delta\Omega_{a_{2k+1}a_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\cdots\wedge\theta_{a_d}\Big)\\
=\;& k\,\epsilon^{a_1\ldots a_d}\,D\delta\omega_{a_1a_2}\wedge\cdots\wedge\Omega_{a_{2k+1}a_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\cdots\wedge\theta_{a_d}\\
=\;& k\,d\Big(\epsilon^{a_1\ldots a_d}\,\delta\omega_{a_1a_2}\wedge\cdots\wedge\Omega_{a_{2k+1}a_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\cdots\wedge\theta_{a_d}\Big)\,. \qquad \text{(A.19.3)}
\end{aligned}
$$

The result is thus established. □

We emphasise that (A.19.2)-(A.19.3) hold *for any variations of the connection*, whether associated with a variation of the frame coefficients or else. This implies that the integral of $\mathscr{L}_k$ over a compact manifold does not depend upon the metric when $2k+2=d$. Since any two Riemannian metric g_1 and g_2 on a manifold M can be joined together by the family of Riemannian metric $tg_1+(1-t)g_2$, $t\in[0,1]$, we conclude that:

PROPOSITION A.19.4 *Let (M,g) be a compact Riemannian manifold of even dimension without boundary, then the integral*

$$\int_M \mathscr{L}_{\frac{d-2}{2}} \qquad \text{(A.19.4)}$$

is metric-independent.

In particular the integral (A.19.4) is an invariant of the differentiable structure of M.

Equation (A.19.9) below shows that Proposition A.19.4 with $d=2$ is closely related to the Gauss-Bonnet theorem,

$$\int_M R = 4\pi\chi(M)\equiv 8\pi(1-g)\,, \qquad \text{(A.19.5)}$$

where g is genus and $\chi(M)$ is the Euler characteristic of M. Proposition A.19.4 fails, however, to convey the whole strength of (A.19.5), as it does not relate the integral to the genus.

In order to determine the field equations arising from the Lagrangeans of Theorem A.19.1 one needs to calculate

$$\frac{\delta\mathscr{L}_k}{\delta g^{ij}} \qquad \text{(A.19.6)}$$

with $2k+2<d$.

If we agree that $\mathscr{L}_{-1}$ is the volume form,

$$\mathscr{L}_{-1}=\theta_1\wedge\cdots\wedge\theta_d=\sqrt{|\det g|}\,dx^1\wedge\cdots\wedge dx^d\,, \qquad \text{(A.19.7)}$$

then

$$\frac{\delta\mathscr{L}_{-1}}{\delta g^{ij}}=\frac{1}{2}g_{ij}\,\sqrt{|\det g|}\,dx^1\wedge\cdots\wedge dx^d\,. \qquad \text{(A.19.8)}$$

This expresses the well known fact that a multiple of the volume form in the action contributes a cosmological-constant term to the field equations.

Equation (A.19.9) with $k = 0$ requires $d \geq 2$ and reads

$$\begin{aligned}
\mathscr{L}_0 &= \epsilon^{a_1\dots a_d}\Omega_{a_1a_2} \wedge \theta_{a_3} \wedge \cdots \wedge \theta_{a_d} \\
&= \frac{1}{2}\epsilon^{a_1\dots a_d} R_{a_1a_2}{}^{b_1b_2}\theta_{b_1} \wedge \theta_{b_2} \wedge \theta_{a_3} \wedge \cdots \wedge \theta_{a_d} \\
&= \frac{1}{2}\epsilon^{a_1\dots a_d}\epsilon_{b_1b_2a_3\dots a_d} R_{a_1a_2}{}^{b_1b_2}\,\theta_1 \wedge \cdots \wedge \theta_d \\
&= (d-2)!\,\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2} R_{a_1a_2}{}^{b_1b_2}\theta_1 \wedge \cdots \wedge \theta_d \\
&= (d-2)!\,R\,\theta_1 \wedge \cdots \wedge \theta_d \\
&= (d-2)!\,R\sqrt{|\det g|}\,dx^1 \wedge \cdots \wedge dx^d
\end{aligned} \tag{A.19.9}$$

where, as usual, R is the scalar curvature. The corresponding contribution to the field equations is the Einstein tensor:

$$\frac{\delta \mathscr{L}_0}{\delta g^{ij}} \sim G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij}\,. \tag{A.19.10}$$

In general the calculation of $\delta\mathscr{L}_k/\delta g_{ij}$ might appear to be tricky because of the constraints

$$0 = \delta g^{ab} = \delta(g^{ij}\theta^a{}_i\theta^b{}_j)\,,\ 0 = \delta T^a = \delta(d\theta^a + \omega^a{}_b \wedge \theta^b)\,,\ 0 = \delta(\omega_{ab} + \omega_{ba})\,, \tag{A.19.11}$$

which have to be satisfied in an orthonormal frame formalism when the curvature tensor arises from the Levi-Civita connection of a metric g. Fortunately, it turns out that for Lagrangeans as in Theorem A.19.1 the equations

$$\frac{\delta \mathscr{L}}{\delta g^{ij}} = 0 \tag{A.19.12}$$

are equivalent to the equations

$$\frac{\partial \mathscr{L}}{\partial \theta^a{}_j} = 0\,, \tag{A.19.13}$$

where the $\theta^a{}_j$ are unconstrained variables, with the $\Omega^a{}_b$'s treated as if independent of the $\theta^a{}_i$'s. The key fact for this is (A.19.3) together with *invariance of* $\mathscr{L}_k$ *under Lorentz transformations.*

To see the equivalence of (A.19.12) and (A.19.13),[4] note first that the last constraint in (A.19.11) is trivial to implement by restricting oneself to anti-symmetric ω_{ab}'s.

Next, the identity (A.19.3) holds as long as the field configuration has vanishing torsion, regardless of whether the variation of the fields satisfies $\delta T^a = 0$. So, the vanishing of the variation of the torsion is irrelevant for the problem at hand.

In order to implement the first constraint in (A.19.11) let us write

$$\delta\theta_a = \frac{1}{2}\Big(\alpha_{ab} + \sigma_{ab}\Big)\theta^b \quad \Longleftrightarrow \quad \delta\theta^a{}_j = \frac{1}{2}\Big(\alpha^a{}_b + \sigma^a{}_b\Big)\theta^b{}_j =: \frac{1}{2}\alpha^a{}_j + \frac{1}{2}\sigma^b{}_j\,, \tag{A.19.14}$$

[4] I am grateful to Orlando Alvarez for pointing out this argument.

where $\alpha_{ab} = \alpha_{ba}$ and $\sigma_{ab} = -\sigma_{ba}$. Then the variations of the fields generated by the σ_{ab}'s correspond to Lorentz-rotations. From (A.19.14) and anti-symmetry of the σ_{ab}'s we have

$$\delta g_{ij} = \delta(g_{ab}\theta^a{}_i\theta^b{}_j) = \alpha_{ab}\theta^a{}_i\theta^b{}_j\,, \tag{A.19.15}$$

so that the α_{ab}'s are in one-to-one correspondence with variations of the metric.

Lorentz-invariance of $\mathscr{L}_k$ implies that for variations with compactly supported σ_{ab}'s and vanishing α_{ab}'s we have, taking into account (A.19.3),

$$0 = \delta\int\mathscr{L}_k = \frac{1}{2}\int\frac{\delta\mathscr{L}_k}{\delta\theta^a{}_j}\sigma^a{}_j = \frac{1}{2}\int\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_j}\sigma^a{}_j\,. \tag{A.19.16}$$

Hence, for all compactly supported variations $\delta\theta^a$ we have

$$\begin{aligned}\delta\int\mathscr{L}_k &= \int\frac{\delta\mathscr{L}_k}{\delta\theta^a{}_j}\delta\theta^a{}_j = \int\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_j}\delta\theta^a{}_j = \underbrace{\frac{1}{2}\int\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_j}\sigma^a{}_j}_{0} + \frac{1}{2}\int\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_j}\alpha^a{}_j \\ &= \frac{1}{2}\int\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_j}\theta^a{}_k g^{k\ell}\delta g_{\ell j} = \frac{1}{2}\int\frac{\delta\mathscr{L}_k}{\delta\theta^a{}_j}\theta^a{}_k g^{k\ell}\delta g_{\ell j}\,.\end{aligned} \tag{A.19.17}$$

This string of equalities shows that for $\mathscr{L}$'s which are a finite sum of $\mathscr{L}_k$'s it holds that

$$\frac{\delta\mathscr{L}}{\delta g_{\ell j}} = 0 \quad\Longleftrightarrow\quad \frac{\delta\mathscr{L}}{\delta\theta^a{}_j}\theta^a{}_k g^{k\ell} = 0 \quad\Longleftrightarrow\quad \frac{\partial\mathscr{L}}{\delta\theta^a{}_j} = 0\,, \tag{A.19.18}$$

as desired.

For further reference we note that (A.19.16) together with the fact that σ_{ab} is an arbitrary antisymmetric tensor field imply that the tensor field $(\partial\mathscr{L}_k/\partial\theta^a{}_i)g_{i\ell}\theta^a{}_j$ is symmetric:

$$\frac{\partial\mathscr{L}_k}{\partial\theta^a{}_i}g_{i\ell}\theta^a{}_j = \frac{\partial\mathscr{L}_k}{\partial\theta^a{}_i}g_{ij}\theta^a{}_\ell\,. \tag{A.19.19}$$

A.19.2 Lovelock tensors

It is a standard fact in calculus of variations that variations of diffeomorphism-invariant Lagrangeans

$$\mathscr{L} = L\,\theta^1\wedge\cdots\wedge\theta^d\,,$$

depending only upon the metric and a finite number of its derivatives, produce tensors with vanishing divergence:

$$0 = \star\nabla^i\left(\frac{\delta\mathscr{L}}{\delta g^{ij}}\right) = \nabla^i\left(\star\frac{\delta\mathscr{L}}{\delta g^{ij}}\right) \equiv \nabla^i\left(\frac{1}{\sqrt{\det g}}\frac{\delta(L\sqrt{\det g})}{\delta g^{ij}}\right)\,. \tag{A.19.20}$$

Here $\star$ denotes Hodge duality, and we continue to assume that the spacetime dimension equals d. Indeed, variations of the metric associated with the flow of a vector field are of the form $\delta g_{ij} = \nabla_{(i}\xi_{j)}$, so that for Lagrangeans depending only upon the metric and its derivatives (up to any order) we have, after a few integrations by parts,

$$\delta \int \mathscr{L} = \int \frac{\delta \mathscr{L}}{\delta g^{ij}} \delta g_{ij} = \int \frac{\delta \mathscr{L}}{\delta g^{ij}} \nabla_{(i}\xi_{j)}\,, \tag{A.19.21}$$

assuming that ξ is compactly supported. When $\mathscr{L}$ is diffeomorphism-invariant the left-hand side vanishes, and yet another integration by parts gives

$$0 = \delta \int \mathscr{L} = -\int \nabla^i (\frac{\delta \mathscr{L}}{\delta g^{ij}}) \xi_j\,. \tag{A.19.22}$$

Since ξ is arbitrary, (A.19.20) follows.

In particular, variations with respect to the metric of linear combination of the Lovelock Lagrangeans $\mathscr{L}_k$ provide symmetric divergence-free tensors which depend only upon the second derivatives of the metric. In fact, as before we find from (A.19.3) that

$$\delta \int \mathscr{L}_k = \int \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_i} \delta \theta^a{}_i\,, \tag{A.19.23}$$

where the derivative $\frac{\partial \mathscr{L}}{\partial \theta^a{}_j}$ is calculated by viewing the Ω_{ab}'s and the $\theta^a{}_j$'s as independent unconstrained variables. Let us set

$$\delta \theta^a{}_i = -\frac{1}{2} g_{i\ell} \theta^a{}_j \delta g^{\ell j}\,. \tag{A.19.24}$$

Inserting (A.19.24) into (A.19.26) with a variation $\delta g^{\ell j}$ arising from a flow, we find

$$\begin{aligned} 0 &= \int \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_i} g_{i\ell} \theta^a{}_j \delta g^{\ell j} = \int \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_i} g_{i\ell} \theta^a{}_j \nabla^{(\ell}\xi^{j)} \\ &= -\int \nabla^j \Big(\frac{\partial \mathscr{L}_k}{\partial \theta^a{}_i} g_{i\ell} \theta^a{}_j \Big) \xi_\ell\,, \end{aligned} \tag{A.19.25}$$

where we have used (A.19.19). Arbitrariness of ξ implies, as before,

$$\nabla^j \Big(\star \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_i} g_{i\ell} \theta^a{}_j \Big)\,, \tag{A.19.26}$$

providing thus for each $0 < k < (d-2)/2$ a symmetric, divergence-free tensor, which is a homogeneous polynomial of order k in the curvature tensor.

We have therefore proved the "if part" of the following theorem of Lovelock:

THEOREM A.19.5 (Lovelock [190]) *All divergence-free symmetric tensors of valence two which depend only upon the metric and its derivatives up to order two are linear combinations of the metric and of the tensor fields*

$$\star \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_j}\, \theta^a \otimes \partial_j\,, \tag{A.19.27}$$

with $\mathscr{L}_k$ given by (A.19.1). Here, when calculating the derivatives in (A.19.27), the moving frame θ^a and the two-forms Ω_{ab} should be considered as independent, unconstrained variables.

An explicit calculation of the tensors (A.19.27) with $2k+2 \le d$ can be carried out as follows: Let, as usual, $\{e_a = e_a{}^j \partial_j\}$ denote the frame dual to $\{\theta^a = \theta^a{}_i dx^i\}$. We have, denoting by $\delta \mathscr{L}_k$ and $\delta\theta^a = \delta\theta^a{}_j dx^j$ the differentials of $\mathscr{L}_k$ and $\theta^a{}_j$ at remaining variables fixed,

$$\begin{aligned}
\delta \mathscr{L}_k &= \frac{\partial \mathscr{L}_k}{\partial \theta^a{}_j}\delta\theta^a{}_j = (d-2k-2)\times \\
&\quad \epsilon^{a_1\dots a_d}\Omega_{a_1a_2}\wedge\dots\wedge\Omega_{a_{2k+1}a_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\dots\wedge g_{aa_d}\delta\theta^a{}_j \underbrace{dx^j}_{e^j{}_b g^{bc}\theta_c} \\
&= (d-2k-2)\times e^j{}_b g^{bc}\,\delta\theta^a{}_j \times \\
&\quad g_{aa_d}\epsilon^{a_1\dots a_d}\Omega_{a_1a_2}\wedge\dots\wedge\Omega_{a_{2k+1}a_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\dots\wedge\theta_c \\
&=: (d-2k-2)\times e^j{}_b g^{bc}\,\delta\theta^a{}_j \times A(k)_{ac}\times\theta^1\wedge\dots\wedge\theta^d\,. \qquad \text{(A.19.28)}
\end{aligned}$$

For $2k+2<d$, we see that the k-th Lovelock tensor $A(k)$, as normalized above, has frame components $A(k)_{ac}$ equal to

$$\begin{aligned}
A(k)_{ac} &:= \star\Big(g_{aa_d}\epsilon^{a_1\dots a_d}\underbrace{\Omega_{a_1a_2}}_{\frac{1}{2}R_{a_1a_2b_1b_2}\theta^{b_1}\wedge\theta^{b_2}}\wedge\dots\wedge\theta_{a_{2k+3}}\wedge\dots\wedge\theta_c\Big) \\
&= \frac{1}{2^{k+1}}\star\Big(g_{aa_d}\epsilon^{a_1\dots a_d}R_{a_1a_2}{}^{b_1b_2}\dots R_{a_{2k+1}a_{2k+2}}{}^{b_{2k+1}b_{2k+2}}\times \\
&\quad \theta_{b_1}\wedge\theta_{b_2}\wedge\dots\wedge\theta_{b_{2k+1}}\wedge\theta_{b_{2k+2}}\wedge\theta_{a_{2k+3}}\wedge\dots\wedge\theta_c\Big) \\
&= \frac{1}{2^{k+1}}g_{aa_d}\epsilon^{a_1\dots a_d}\epsilon_{b_1\dots b_{2k+2}a_{2k+3}\dots a_{d-1}c}\,R_{a_1a_2}{}^{b_1b_2}\dots R_{a_{2k+1}a_{2k+2}}{}^{b_{2k+1}b_{2k+2}} \\
&= \underbrace{\frac{(d-2k-3)!(2k+3)!}{2^{k+1}}}_{=:c_{k,d}}\,\varepsilon\,\delta^{a_1}_{[b_1}\dots\delta^{a_{2k+2}}_{b_{2k+2}}\delta^{a_d}_{c]}g_{aa_d}R_{a_1a_2}{}^{b_1b_2}\dots R_{a_{2k+1}a_{2k+2}}{}^{b_{2k+1}b_{2k+2}}\,, \quad \text{(A.19.29)}
\end{aligned}$$

with $\varepsilon = 1$ in Riemannian and $\varepsilon = -1$ in Lorentzian signature.

INCIDENTALLY: The following simple proof of symmetry of the tensors $A(k)$ has been pointed out to us by O. Alvarez. Set

$$\eta_{aa_1\dots a_{2k+2}bb_1\dots b_{2k+2}} := g_{ac}g_{a_1c_1}\dots g_{a_{2k+2}c_{2k+2}}\delta^{c_1}_{[b_1}\dots\delta^{c_{2k+2}}_{b_{2k+2}}\delta^{c}_{b]}\,. \qquad \text{(A.19.30)}$$

Clearly

$$\eta_{aa_1\dots a_{2k+2}bb_1\dots b_{2k+2}} = \eta_{bb_1\dots b_{2k+2}aa_1\dots a_{2k+2}}\,. \qquad \text{(A.19.31)}$$

From (A.19.29), (A.19.31) and the pair-interchange symmetry of the Riemann tensor we have

$$\begin{aligned}
A(k)_{ab} &= c_{k,d}\,\varepsilon\,\eta_{aa_1\dots a_{2k+2}bb_1\dots b_{2k+2}}R^{a_1a_2b_1b_2}\dots R^{a_{2k+1}a_{2k+2}b_{2k+1}b_{2k+2}} \\
&= c_{k,d}\,\varepsilon\,\eta_{bb_1\dots b_{2k+2}aa_1\dots a_{2k+2}}R^{a_1a_2b_1b_2}\dots R^{a_{2k+1}a_{2k+2}b_{2k+1}b_{2k+2}} \\
&= c_{k,d}\,\varepsilon\,\eta_{bb_1\dots b_{2k+2}aa_1\dots a_{2k+2}}R^{b_1b_2a_1a_2}\dots R^{b_{2k+1}b_{2k+2}a_{2k+1}a_{2k+2}} \\
&= A(k)_{ba}\,, \qquad \text{(A.19.32)}
\end{aligned}$$

as desired. □

As an example of tensors (A.19.29), when $k = 0$ we have

$$\begin{aligned} A(0)^a{}_c &\sim 3\delta^{a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^a_{c]}R_{a_1a_2}{}^{b_1b_2} = (\delta^{a_1}_{[c}\delta^{a_2}_{b_1]}\delta^a_{b_2} + \delta^{a_1}_{[b_1}\delta^{a_2}_{b_2]}\delta^a_c + \delta^{a_1}_{[b_2}\delta^{a_2}_{c]}\delta^a_{b_1})R_{a_1a_2}{}^{b_1b_2} \\ &= (R_{cb_1}{}^{b_1a} + R_{b_1b_2}{}^{b_1b_2}\delta^a_c + R_{b_2c}{}^{ab_2}) = -2(R^a{}_c - \frac{1}{2}R\delta^a_c)\,, \qquad \text{(A.19.33)} \end{aligned}$$

which is proportional to the Einstein tensor, consistently with (A.19.10).

The case $k = 1$ requires a somewhat lengthy calculation [189]

$$\begin{aligned} A(1)^{ij} &\sim 2R^{iu}R_u{}^j + 2R_{uv}R^{uivj} - RR^{ij} - R^{iuvs}R^j{}_{uvs} \\ &\quad -\frac{1}{4}g^{ij}(4R_{uv}R^{uv} - R^2 - R_{uvst}R^{uvst})\,. \qquad \text{(A.19.34)} \end{aligned}$$

A.20 Clifford algebras

Our approach to the description of *Clifford algebras* is a variation upon [36].

Let q be a *non-degenerate* quadratic form on a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{A}$ be an algebra over $\mathbb{K}$. A *Clifford map* of (W, q) into $\mathscr{A}$ is a linear map $f : W \to \mathscr{A}$ with the property that, for any $X \in W$,

$$f(X)^2 = -q(X, X)\,. \qquad \text{(A.20.1)}$$

By polarisation, this is equivalent to

$$f(X)f(Y) + f(Y)f(X) = -2q(X, Y)\,. \qquad \text{(A.20.2)}$$

for any $X, Y \in W$.

Note that a Clifford map is necessarily injective: if $f(X) = 0$ then $q(X, Y) = 0$ for all Y by (A.20.2), hence $X = 0$. Thus $\dim \mathscr{A} \geq \dim W$ whenever a Clifford map exists.

The *Clifford algebra* $C\ell(W, q)$ is the unique (up to homomorphism) associative algebra with unity defined by the following two properties:

1. there exists a Clifford map $\kappa : W \to C\ell(W, q)$, and

2. for any Clifford map $f : W \to \mathscr{A}$ there exists exactly one homomorphism $\tilde f$ of algebras with unity $\tilde f : C\ell(W, q) \to \mathscr{A}$ such that

$$f = \tilde f \circ \kappa\,.$$

The definition is somewhat roundabout, and takes a while to absorb. The key property is the Clifford anti-commutation rule (A.20.2). The second point is a way of saying that $C\ell(W, q)$ is the smallest algebra for which (A.20.2) holds.

Now, uniqueness of $C\ell(W, q)$, up to algebra homomorphism, follows immediately from its definition. Existence is a consequence of the following construction: let $\mathscr{T}(W)$ be the tensor algebra of W,

$$\mathscr{T}(W) := \mathbb{K} \oplus W \oplus_{\ell=2}^{\infty} \underbrace{W \otimes \ldots \otimes W}_{\ell \text{ factors}}\,,$$

it being understood that only elements with a finite number of non-zero components in the infinite sum are allowed. Then $\mathscr{T}(W)$ is an associative algebra with unity, the product of two elements $a, b \in \mathscr{T}(W)$ being the tensor product $a \otimes b$. Let I_q be the two-sided ideal generated by all tensors of the form $X \otimes X + q(X, X)$, $X \in W$. Then the quotient algebra

$$\mathscr{T}(W)/I_q$$

has the required property. Indeed, let κ be the map which to $X \in W \subset \mathscr{T}(W)$ assigns the equivalence class $[X] \in \mathscr{T}(W)/I_q$. Then κ is a Clifford map by definition. Further, if $f : V \to \mathscr{A}$ is a Clifford map, let $\hat{f}$ be the unique linear map $\hat{f} : \mathscr{T}(W) \to \mathscr{A}$ satisfying

$$\hat{f}(X_1 \otimes \ldots \otimes X_k) = f(X_1) \cdots f(X_k) \,.$$

Then $\hat{f}$ vanishes on I_q, hence provides the desired map $\tilde{f}$ defined on the quotient, $\tilde{f}([X]) := \hat{f}(X)$.

Example A.20.1 Let $W = \mathbb{R}$, with $q(x) = x^2$. Then $\mathbb{C}$ with $\kappa(x) = xi$, satisfies the Clifford product rule. Clearly (A.20.1) cannot be satisfied in any smaller algebra, so (up to homomorphism) $C\ell(W, q) = \mathbb{C}$.

Example A.20.2 Let $W = \mathbb{R}$, with $q'(x) = -x^2$. Then $C\ell(W, q') = \mathbb{R}$, $\kappa(x) = x$. Comparing with Example A.20.1, one sees that passing to the opposite signature matters.

Example A.20.3 Consider the hermitian, traceless, *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.20.3}$$

and set $\sigma^i = \sigma_i$. One readily checks that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma^k \qquad \Longrightarrow \qquad \{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \,, \tag{A.20.4}$$

where for any two matrices a, b the *anti-commutator* $\{a, b\}$ is defined as

$$\{a, b\} = ab + ba \,.$$

Hence, for any $X \in \mathbb{R}^3$ it holds that

$$(X^k i\sigma_k)(X^\ell i\sigma_\ell) = -\delta(X, X) \,,$$

where δ is the standard scalar product on $W = \mathbb{R}^3$. Thus, the map

$$X = (X^k) \to X^k i\sigma_k$$

is a Clifford map on $(\mathbb{R}^3, \delta)$, and in fact the algebra generated by the matrices $\gamma_k := i\sigma_k$ is homomorphic to $C\ell(\mathbb{R}^3, \delta)$. This follows again from the fact that no smaller dimension is possible.

EXAMPLE A.20.4 Let σ_i be the Pauli matrices (A.20.3) and let the 4×4 complex valued matrices be defined as

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{C}^2} \\ \mathrm{id}_{\mathbb{C}^2} & 0 \end{pmatrix} = -\gamma_0 \,, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \gamma^i \,. \tag{A.20.5}$$

We note that γ_0 is hermitian, while the γ_i's are anti-hermitian with respect to the canonical hermitian scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $\mathbb{C}^4$. From Equation (A.20.5) one immediately finds

$$\{\gamma_i,\gamma_j\} = \begin{pmatrix} -\{\sigma_i,\sigma_j\} & 0 \\ 0 & -\{\sigma_i,\sigma_j\} \end{pmatrix}, \quad \{\gamma_i,\gamma_0\} = 0\,, \quad (\gamma_0)^2 = 1\,,$$

and (A.20.4) leads to the Clifford product relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab} \tag{A.20.6}$$

for the Minkowski metric $g_{ab} = \mathrm{diag}(-1,1,1,1)$.

A real representation of the commutation relations (A.20.6) on $\mathbb{R}^8$ can be obtained by viewing $\mathbb{C}^4$ as a vector space over $\mathbb{R}$, so that 1) each 1 above is replaced by $\mathrm{id}_{\mathbb{R}^2}$, and 2) each i is replaced by the antisymmetric 2×2 matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

More precisely, let us define the 4×4 matrices $\hat\sigma_i$ by

$$\hat\sigma_1 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^2)} & \mathrm{id}_{\mathbb{R}^2} \\ \mathrm{id}_{\mathbb{R}^2} & 0_{\mathrm{End}(\mathbb{R}^2)} \end{pmatrix}, \quad \hat\sigma_3 = \begin{pmatrix} \mathrm{id}_{\mathbb{R}^2} & 0_{\mathrm{End}(\mathbb{R}^2)} \\ 0_{\mathrm{End}(\mathbb{R}^2)} & -\mathrm{id}_{\mathbb{R}^2} \end{pmatrix}, \tag{A.20.7}$$

$$\hat\sigma_2 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^2)} & -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0_{\mathrm{End}(\mathbb{R}^2)} \end{pmatrix}, \tag{A.20.8}$$

which are clearly symmetric, and the new γ's by

$$\gamma^0 = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^4)} & \mathrm{id}_{\mathbb{R}^4} \\ \mathrm{id}_{\mathbb{R}^4} & 0_{\mathrm{End}(\mathbb{R}^4)} \end{pmatrix} = -\gamma_0 \,, \qquad \gamma_i = \begin{pmatrix} 0_{\mathrm{End}(\mathbb{R}^4)} & \hat\sigma_i \\ -\hat\sigma_i & 0_{\mathrm{End}(\mathbb{R}^4)} \end{pmatrix} = \gamma^i \,. \tag{A.20.9}$$

It should be clear that the γ's satisfy (A.20.6), with γ_0 symmetric, and γ_i's - antisymmetric.

Let us return to general considerations. Choose a basis e_i of W, and consider any element $a \in \mathscr{T}(W)$. Then a can be written as

$$a = \alpha + \sum_{k=1}^{N} \sum_{i_1,\ldots,i_k} a^{i_1\ldots i_k} e_{i_1}\otimes\ldots\otimes e_{i_k}\,,$$

for some N depending upon a. When passing to the quotient, every tensor product $e_{i_j}\otimes e_{i_r}$ with $i_j > i_r$ can be replaced by $-2g_{i_j i_r} - e_{i_r}\otimes e_{i_j}$, leaving eventually only those indices which are increasingly ordered. Thus, a is equivalent to

$$\beta + b^i e_i + \sum_{k=N}^{k} \sum_{i_1<\ldots<i_k} b^{i_1\ldots i_k} e_{i_1}\otimes\ldots\otimes e_{i_k}\,,$$

for some new coefficients. For example

$$\begin{aligned}\alpha + a^i e_i + a^{ij} e_i\otimes e_j &= \alpha + a^i e_i + a^{ij}(\underbrace{e_{(i}\otimes e_{j)}}_{\sim -q(e_i,e_j)} + e_{[i}\otimes e_{j]})\\ &\sim \alpha - a^{ij} q(e_i,e_j) + a^i e_i + a^{ij} e_{[i}\otimes e_{j]}\,.\end{aligned}$$

This implies that elements of the form

$$\gamma_{i_1\ldots i_k} := [e_{i_1}\otimes\ldots\otimes e_{i_k}]\,,\quad i_1<\ldots<i_k$$

span $C\ell(W,q)$. (Here the outermost bracket is the equivalence relation in $\mathscr{T}(W)$.) Equivalently,

$$C\ell(W,q) = \mathbb{K}\oplus \mathrm{Vect}\{\gamma_{i_1\ldots i_k}\}\,,\qquad \text{where } \gamma_i := \kappa(e_i)\,,\ \gamma_{i_1\ldots i_k} := \gamma_{[i_1}\cdots\gamma_{i_k]}\,.$$

We conclude that the dimension of $C\ell(W,q)$ is less than or equal to that of the exterior algebra of W, in particular $C\ell(W,q)$ is finite dimensional (recall that it was part of our definition that $\dim W < \infty$). The reader is warned that the above elements of the algebra are not necessarily linearly independent, as can be seen in Examples A.20.2 and A.20.3.

It should be clear to the reader that the linear map, which is deduced by the considerations above, from the exterior algebra to the Clifford algebra, does *not* preserve the product structures in those algebras.

A *representation* (V,ρ) of a Clifford algebra $C\ell(W,q)$ on a vector space V over $\mathbb{K}$ is a map $\rho : C\ell(W,q)\to \mathrm{End}(V)$ such that $\rho\circ\kappa$ is a Clifford map. It immediately follows from the definition of the Clifford algebra that ρ is uniquely defined by its restriction to $\kappa(W)$.

A fundamental fact is the following:

PROPOSITION A.20.5 *Let q be positive definite and let (V,ρ) be a representation of $C\ell(W,q)$. If $\mathbb{K}=\mathbb{R}$, then there exists a scalar product $\langle\cdot,\cdot\rangle$ on V so that $\rho\circ\kappa(X)$ is antisymmetric for all $X\in W$. Similarly if $\mathbb{K}=\mathbb{C}$, then there exists a hermitian product $\langle\cdot,\cdot\rangle$ on V so that $\rho\circ\kappa(X)$ is antihermitian for all $X\in W$.*

PROOF: Let e_i be any basis of W, set $\gamma_i := \rho(\kappa(e_i))$, since ρ is a representation the γ_i's satisfy the relation (A.20.6). Let γ_I run over the set

$$\Omega := \{\pm 1\,, \pm\gamma_i\,, \pm\gamma_{i_1\ldots i_k}\}_{1\le i_1<\cdots<i_k\le n}\,.$$

It is easy to check that $\Omega\gamma_i\subset\Omega$, but since

$$(\Omega\gamma_i)\gamma_i = \Omega\underbrace{\gamma_i\gamma_i}_{=-1} = -\Omega = \Omega\,,$$

we conclude that $\Omega\gamma_i = \Omega$. Let $(\cdot,\cdot)$ denote any scalar product on V, and for $\psi, \varphi \in V$ set

$$\langle\psi,\varphi\rangle := \sum_{\gamma_I\in\Omega} (\gamma_I\psi, \gamma_I\varphi)\,.$$

Then for any $1 \le \ell \le n$ we have (no summation over ℓ)

$$\begin{aligned}
\langle(\gamma_\ell)^t\gamma_\ell\psi,\varphi\rangle &= \langle\gamma_\ell\psi,\gamma_\ell\varphi\rangle = \sum_{\gamma_I\in\Omega}(\gamma_I\gamma_\ell\psi,\gamma_I\gamma_\ell\varphi)\\
&= \sum_{\gamma_I\in\Omega\gamma_\ell}(\gamma_I\psi,\gamma_I\varphi) = \sum_{\gamma_I\in\Omega}(\gamma_I\psi,\gamma_I\varphi)\\
&= \langle\psi,\varphi\rangle\,.
\end{aligned}$$

Since this holds for all ψ and φ we conclude that $(\gamma_\ell)^t\gamma_\ell = \mathrm{Id}$. Multiplying from the right with γ_ℓ, and recalling that $(\gamma_\ell)^2 = -\mathrm{Id}$ we obtain $(\gamma_\ell)^t = -\gamma_\ell$. Now, by definition,

$$(\rho\circ\kappa(X))^t = (X^a\gamma_a)^t = -X^a\gamma_a = -\rho\circ\kappa(X)\,,$$

as desired.

An identical calculation applies for hermitian scalar products. □

Incidentally: The scalar product constructed above is likely to depend upon the initial choice of basis e_a, but this is irrelevant for the problem at hand, since the statement that $\rho\circ\kappa(X)$ is anti-symmetric, or anti-hermitian, is basis-independent. □

Throughout most of this work, when q is positive definite we will only use scalar products on V for which the representation of $C\ell(W,q)$ is anti-symmetric or anti-hermitian.

Let $\dim V > 0$. A representation (V,ρ) of $C\ell(W,q)$ is said to be *reducible* if V can be decomposed as a direct sum $V_1 \oplus V_2$ of nontrivial subspaces, each of them being invariant under all maps in $\rho(C\ell(W,q))$. The representation (V,ρ) is said to be *irreducible* if it is not reducible. Note that the existence of an invariant space does not *a priori* imply the existence of a complementing space which is invariant as well. However, we have the following:

Proposition A.20.7 *Every finite dimensional representation*

$$\rho : C\ell(W,q) \to End(V)$$

of $C\ell(W,q)$ such that V contains a non-trivial invariant subspace is reducible. Hence, $V = \oplus_{i=1}^k V_i$, $\rho = \oplus_{i=1}^k \rho_i$, with $(V_i, \rho_i := \rho|_{V_i})$ irreducible.

Proof: Suppose that there exists a subspace $V_1 \subset V$ invariant under ρ. We can assume that V_1 has no invariant subspaces, otherwise we pass to this subspace and call it V_1; in a finite number of steps we obtain a subspace V_1 such that $\rho|_{V_1}$ is irreducible. The proof of Proposition A.20.5 provides a scalar or hermitian product $\langle\cdot,\cdot\rangle$ on V which is invariant under the action of all maps in Ω. Then $V = V_1 \oplus (V_1)^\perp$, and it is easily checked that $(V_1)^\perp$ is also invariant under maps in Ω, hence under all maps in the image of ρ. One can repeat now the whole argument with V replaced by $(V_1)^\perp$, and the claimed decomposition is obtained after a finite number of steps. □

It is sometimes convenient to use irreducible representations, which involves no loss of generality in view of Proposition A.20.7. However, we will not assume irreducibility unless explicitly specified otherwise.

REMARK A.20.8 According to Trautman [266], there exist two inequivalent irreducible representations in odd space-dimension, the one related to the other by the map $X \mapsto -X$. □

A.20.1 Eigenvalues of γ-matrices

In the proofs of the energy-momentum inequalities the positivity properties of several matrices acting on the space of spinors have to be analysed. It is sufficient to make a pointwise analysis, so we consider a real vector space V equipped with a scalar product $\langle \cdot, \cdot \rangle$ together with matrices γ_μ, $\mu = 0, 1, \cdots, n$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\eta_{\mu\nu} , \qquad (A.20.10)$$

where $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$. We further suppose that the matrices γ_μ^t, transposed with respect to $\langle \cdot, \cdot \rangle$, satisfy

$$\gamma_0^t = \gamma_0 , \quad \gamma_i^t = -\gamma_i ,$$

where the index i runs from one to n. Let us start with

$$a^\mu \gamma_0 \gamma_\mu = a^0 + a^i \gamma_0 \gamma_i , \quad (a^\mu) = (a^0, \vec{a}) = (a^0, (a^i)) .$$

The matrices $a^i \gamma_0 \gamma_i$ are symmetric and satisfy

$$(a^i \gamma_0 \gamma_i)^2 = a^i a^j \gamma_0 \gamma_i \gamma_0 \gamma_j = -a^i a^j \gamma_0 \gamma_0 \gamma_i \gamma_j = |\vec{a}|_\delta^2 ,$$

so that the eigenvalues belong to the set $\{\pm |\vec{a}|_\delta\}$. Since γ_0 anticommutes with $a^i \gamma_0 \gamma_i$, it interchanges the eigenspaces with positive and negative eigenvalues. Let ψ_i, $i = 1, \ldots, N$, be an ON basis of the $|\vec{a}|_\delta$ eigenspace of $a^i \gamma_0 \gamma_i$, set

$$\phi_{2i-1} = \psi_i , \quad \phi_{2i} = \gamma_0 \psi_i .$$

It follows that $\{\phi_i\}_{i=1}^{2N}$ forms an ON basis of V (in particular $\dim V = 2N$), and in that basis $a^\mu \gamma_0 \gamma_\mu$ is diagonal with entries $a^0 \pm |\vec{a}|_\delta$. We have thus proved

PROPOSITION A.20.9 *The quadratic form $\langle \psi, a^\mu \gamma_0 \gamma_\mu \psi \rangle$ is non-negative if and only if $a^0 \geq |\vec{a}|_\delta$.*

Let us consider, next, the symmetric matrix

$$A := a^\mu \gamma_0 \gamma_\mu + b\gamma_0 + c\gamma_1 \gamma_2 \gamma_3 . \qquad (A.20.11)$$

Let ψ_1 be an eigenvector of $a^i \gamma_0 \gamma_i$ with eigenvalue $|\vec{a}|_\delta$, set

$$\phi_1 = \psi_1 , \quad \phi_2 = \gamma_0 \psi_1 , \quad \phi_3 = \gamma_1 \gamma_2 \gamma_3 \psi_1 , \quad \phi_4 = \gamma_1 \gamma_2 \gamma_3 \gamma_0 \psi_1 .$$

From the commutation relations (A.20.10) one easily finds

$$a^i\gamma_0\gamma_i\phi_1 = |\vec{a}|_\delta\phi_1\,, \quad a^i\gamma_0\gamma_i\phi_2 = -|\vec{a}|_\delta\phi_2\,, \quad a^i\gamma_0\gamma_i\phi_3 = -|\vec{a}|_\delta\phi_3\,, \quad a^i\gamma_0\gamma_i\phi_4 = |\vec{a}|_\delta\phi_4\,,$$

$$\gamma_0\phi_1 = \phi_2\,, \quad \gamma_0\phi_2 = \phi_1\,, \quad \gamma_0\phi_3 = -\phi_4\,, \quad \gamma_0\phi_4 = -\phi_3\,,$$

$$\gamma_1\gamma_2\gamma_3\phi_1 = \phi_3\,, \quad \gamma_1\gamma_2\gamma_3\phi_2 = \phi_4\,, \quad \gamma_1\gamma_2\gamma_3\phi_3 = \phi_1\,, \quad \gamma_1\gamma_2\gamma_3\phi_4 = \phi_2\,.$$

It is simple to check that the ϕ_i's so defined are ON; proceeding by induction one constructs an ON-basis $\{\phi_i\}_{i=1}^{2N}$ of V (in particular $\dim V$ is a multiple of 4) in which A is block-diagonal, built-out of blocks of the form

$$\begin{pmatrix} a^0+|\vec{a}|_\delta & b & c & 0 \\ b & a^0-|\vec{a}|_\delta & 0 & c \\ c & 0 & a^0-|\vec{a}|_\delta & -b \\ 0 & c & -b & a^0+|\vec{a}|_\delta \end{pmatrix}.$$

The eigenvalues of this matrix are easily found to be $a^0 \pm \sqrt{|\vec{a}|_\delta^2 + b^2 + c^2}$. We thus have:

Proposition A.20.10 *We have the sharp inequality*

$$\langle\psi, (a^\mu\gamma_0\gamma_\mu + b\gamma_0 - c\gamma_1\gamma_2\gamma_3)\,\psi\rangle \geq \left(a^0 - \sqrt{|\vec{a}|_\delta^2 + b^2 + c^2}\right)|\psi|^2\,,$$

in particular the quadratic form $\langle\psi, A\psi\rangle$, with A defined in (A.20.11), is non-negative if and only if

$$a^0 \geq \sqrt{|\vec{a}|_\delta^2 + b^2 + c^2}\,.$$

A.21 Killing vectors and isometries

Let (M,g) be a pseudo-Riemannian manifold. A map ψ is called an *isometry* if

$$\psi^*g = g\,, \tag{A.21.1}$$

where ψ^* is the pull-back map defined in Section A.8.2.

A standard fact is that the group $\mathrm{Iso}(M,g)$ of isometries of (M,g) carries a natural manifold structure. Such groups, when non-discrete, are called *Lie groups*. If (M,g) is Riemannian and compact, then $\mathrm{Iso}(M,g)$ is compact.

It is also a standard fact that any element of the connected component of the identity of a Lie group G belongs to a one-parameter subgroup $\{\phi_t\}_{t\in\mathbb{R}}$ of G. This allows one to study actions of isometry groups by studying the generators of one-parameter subgroups, defined as

$$X(f)(x) = \left.\frac{d(f(\phi_t(x)))}{dt}\right|_{t=0} \qquad \Longleftrightarrow \qquad X = \left.\frac{d\phi_t}{dt}\right|_{t=0}.$$

The vector fields X obtained in this way are called *Killing vectors*. The knowledge of Killing vectors provides considerable amount of information on the isometry group, and we thus continue with an analysis of their properties. We will see shortly that the collection of Killing vectors forms a Lie algebra: by definition, a *Lie algebra* is a vector space equipped with a bracket operation such that

$$[X,Y] = -[Y,X]\,,$$

and

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y] = 0\,.$$

In the case of Killing vectors, the bracket operation will be the usual bracket of vector fields.

Of key importance to us will be the fact, that the dimension of the isometry group of $(\mathscr{M},g)$ equals the dimension of the space of the Killing vectors.

A.21.1 Killing vectors

Let ϕ_t be a one-parameter group of isometries of $(\mathscr{M},g)$, thus

$$\phi_t^* g = g \quad \Longrightarrow \quad \mathscr{L}_X g = 0\,. \tag{A.21.2}$$

Recall that (see (A.8.7), p. 239)

$$\mathscr{L}_X g_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\alpha g_{\mu\alpha}\,.$$

In a coordinate system where the partial derivatives of the metric vanish at a point p, the right-hand side equals $\nabla_\mu X_\nu + \nabla_\nu X_\mu$. But the left-hand side is a tensor field, and two tensor fields equal in one coordinate system coincide in all coordinate systems. We have thus proved that generators of isometries satisfy the equation

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0\,. \tag{A.21.3}$$

Conversely, consider a solution of (A.21.3); any such solution is called a *Killing vector*. From the calculation just carried out, the Lie derivative of the metric with respect to X vanishes. This means that the local flow of X preserves the metric. In other words, X generates local isometries of g.

To make sure that X generates a one-parameter group of isometries one needs moreover to make sure that X is *complete*. By definition, this means that the integral curves of X, i.e. solutions of

$$\frac{dx}{dt} = X(x(t))\,, \quad x(0) = x_0\,, \tag{A.21.4}$$

are defined for all values of parameter $t \in \mathbb{R}$ for all initial points x_0. This might be difficult to establish, often requiring further global hypotheses; we return to this in Appendix A.21.4. The map $(t,x_0) \mapsto x(t)$, where $x(t)$ is the solution of (A.21.4), is often denoted by $\phi_t(x_0)$, and is called the flow of X. We will sometimes write $\phi_t[X]$ when more than one vector X is involved.

Recall the identity (A.8.8), p. 239:

$$\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]\,. \tag{A.21.5}$$

This implies that the commutator of two Killing vector fields is a Killing vector field:

$$\mathscr{L}_{[X,Y]}g = \mathscr{L}_X(\underbrace{\mathscr{L}_Y g}_{0}) - \mathscr{L}_Y(\underbrace{\mathscr{L}_X g}_{0}) = 0\,.$$

Thus, and as already pointed-out, the collection of all Killing vector fields, equipped with the Lie bracket, forms a *Lie algebra*.

REMARK A.21.1 Let (M,g) be a complete Riemannian manifold, than all Killing vector fields are complete. To see this, let ϕ_t be generated by a Killing vector X, let $p \in M$ and let $\gamma(t) = \phi_t(p)$ be the integral curve of X through p, thus $\dot\gamma = X$. We claim, first, that the length of X is preserved along the orbits of X. Indeed:

$$\frac{d(X^iX_i)}{dt} = 2X^kX^i\nabla_kX_i = 0\,,$$

as ∇_kX_i is antisymmetric. Next, the length of any segment of $\gamma(t)$ equals to

$$\int_{t_1}^{t_2} |\dot\gamma|dt = \int_{t_1}^{t_2} |X|dt = |X(p)|(t_2-t_1)\,.$$

Hence, any integral curve of the flow defined on a bounded interval of parameters has finite length. The fact that $\gamma(t)$ is defined for all t follows now from completeness of (M,g) by simple considerations. □

REMARK A.21.2 Let p be a fixed point of an isometry ϕ. Then ϕ_* maps T_pM to T_pM; we will refer to this action as the *tangent action*.

For $W \in T_pM$ let $s \mapsto \gamma_W(s)$ be an affinely parameterised geodesic with $\gamma_W(0) = p$ and $\dot\gamma(0) = W$. Since isometries map geodesics to geodesics, the curve $s \mapsto \phi(\gamma_W(s))$ is a geodesic that passes through p and has tangent vector ϕ_*W there. As the affine parameterisation condition is also preserved by isometries, we conclude that

$$\phi(\gamma_W(s)) = \gamma_{\phi_*W}(s)\,. \tag{A.21.6}$$

In particular, in the Riemannian case ϕ maps the metric spheres and balls

$$S_p(r) := \{q \in M :\ d(p,q) = r\}\,, \quad B_p(r) := \{q \in M :\ d(p,q) \le r\}$$

to themselves. Similarly the $S_p(r)$'s and $B_p(r)$'s are invariant in the Lorentzian case as well, or for that matter in any signature, but these sets are not topological spheres or topological balls anymore.

The action of a group of transformations is called *transitive* if for every pair $p,q \in M$ there exist an element ϕ of the group such that $q = \phi(p)$. Suppose that the tangent action on T_pM, of those elements of $\mathrm{Iso}(M,g)$ which leave p fixed, is transitive on unit vectors (this is only possible for Riemannian metrics, since isometries preserve the causal nature of vectors). What we just said shows that, for complete Riemannian metrics, transitivity on unit vectors at p implies that the action on the $S_p(r)$'s is transitive as well. □

REMARK A.21.3 Let p be a point in a three-dimensional Riemannian manifold (M,g) such that the tangent action of $\mathrm{Iso}(M,g)$ is transitive on unit vectors of

T_pM. The group of isometries of M that leave p fixed is then a closed subgroup of $SO(3)$ which acts transitively on S^2, hence of dimension at least two. Now, it is easily seen (exercice) that connected subgroups of $SO(3)$ are $\{e\}$ (which has dimension zero), $U(1)$ (which has dimension one), or $SO(3)$ itself. We conclude that existence of fixed points of the action implies that the group of isometries of (M, g) contains an $SO(3)$ subgroup. □

Remark A.21.4 In Riemannian geometry, the *sectional curvature* κ of a plane spanned by two vectors $X, Y \in T_pM$ is defined as

$$\kappa(X,Y) := \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}\,. \tag{A.21.7}$$

A simple calculation shows that κ depends only upon the plane, and not the choice of the vectors X and Y spanning the plane. The definition extends to pseudo-Riemannian manifolds as long as the denominator does not vanish; equivalently, the plane spanned by X and Y should *not* be null.

For maximally symmetric Riemannian manifolds the action of the isometry group on the collection of two-dimensional subspaces of the tangent bundle is transitive, which implies that κ is independent of p. *Complete* Riemannian manifolds with *constant* κ, not necessarily simply connected, are called *space forms.* □

Remark A.21.5 A *complete Riemannian* manifold (M, g) which is isotropic *around every point* is necessarily homogeneous. To see this, let $p, p' \in M$, and let q be any point such that the distance from q to p equals that from q to p', say r. Then both p and p' lie on the distance sphere $S(q, r)$, and since (M, g) is isotropic at q, it follows from Remark A.21.2 that there exists an isometry which leaves q fixed and which maps p into p'. □

Equation (A.21.3) leads to a second order system of equations, as follows: Taking cyclic permutations of the equation obtained by differentiating (A.21.3) one has

$$\begin{aligned}
-\nabla_\gamma\nabla_\alpha X_\beta - \nabla_\gamma\nabla_\beta X_\alpha &= 0\,,\\
\nabla_\alpha\nabla_\beta X_\gamma + \nabla_\alpha\nabla_\gamma X_\beta &= 0\,,\\
\nabla_\beta\nabla_\gamma X_\alpha + \nabla_\beta\nabla_\alpha X_\gamma &= 0\,.
\end{aligned}$$

Adding, and expressing commutators of derivatives in terms of the Riemann tensor, one obtains

$$\begin{aligned}
2\nabla_\alpha\nabla_\beta X_\gamma &= (R_{\sigma\gamma\beta\alpha} + R_{\sigma\alpha\beta\gamma} + \underbrace{R_{\sigma\beta\alpha\gamma}}_{=-R_{\sigma\alpha\gamma\beta}-R_{\sigma\gamma\beta\alpha}})X^\sigma\\
&= 2R_{\sigma\alpha\beta\gamma}X^\sigma\,.
\end{aligned}$$

Thus

$$\nabla_\alpha\nabla_\beta X_\gamma = R_{\sigma\alpha\beta\gamma}X^\sigma\,. \tag{A.21.8}$$

Example A.21.6 As an example of application of (A.21.8), let (M, g) be flat. In a coordinate system $\{x^\mu\}$ in which the metric has constant entries (A.21.8) reads

$$\partial_\alpha\partial_\beta X_\gamma = 0\,.$$

The solutions are therefore linear,

$$X^\alpha = A^\alpha + B^\alpha{}_\beta x^\beta \,.$$

Plugging this into (A.21.3), one finds that $B_{\alpha\beta}$ must be anti-symmetric. Hence, the dimension of the set of all Killing vectors of $\mathbb{R}^{n,m}$, and thus of $\mathrm{Iso}(\mathbb{R}^{n,m})$, is $(n+m)(n+m+1)/2$, independently of signature.

Consider, next, a torus $\mathbb{T}^n := S^1 \times \ldots \times S^1$, equipped again with a flat metric. We claim that none of the locally defined Killing vectors of the form $B^i{}_j x^j$ survive the periodic identifications, so that the dimension of $\mathrm{Iso}(\mathbb{T}^n, \delta)$ is n: Indeed, using (A.21.8) and integration by parts we have

$$\int X^i \underbrace{D_j D_i X^j}_{=0} = -\int D^j X^i \underbrace{D_i X_j}_{=-D_j X_i} = \int |DX|^2 \,, \tag{A.21.9}$$

and so $B_{ij} \equiv D_i X_j = 0$: all Killing vectors on a flat Riemannian $\mathbb{T}^n$ are covariantly constant.

In fact, an obvious modification of the last calculation shows that *the isometry group of a compact Riemannian manifold with strictly negative Ricci tensor is finite*, and that *non-trivial Killing vectors of compact Riemannian manifolds with non-positive Ricci tensor are covariantly constant.* Indeed, for such manifolds the left-hand side of (A.21.9) does not necessarily vanish a priori, and instead we have

$$\int |DX|^2 = \int X^i D_j D_i X^j = \int X^i R_{kji}{}^j X^j = \int X^i R_{ki} X^j \,, \tag{A.21.10}$$

The left-hand side is always positive. If the Ricci tensor is non-positive, then the right-hand side is non-positive, which is only possible if both vanish, hence $DX = 0$ and $R_{ij} X^i X^j = 0$. If the Ricci tensor is strictly negative, then $X = 0$, and there are no non-trivial Kiling vectors, so that the dimension of the group of isometries is zero. Since the group is compact when (M, g) is Riemannian and compact, it must be finite when no Killing vectors exist. □

An important consequence of (A.21.8) is:

PROPOSITION A.21.7 *Let M be connected and let $p \in M$. A Killing vector is uniquely defined by its value $X(p)$ and the value at p of the anti-symmetric tensor $\nabla X(p)$.*

PROOF: Consider two Killing vectors X and Y such that $X(p) = Y(p)$ and $\nabla X(p) = \nabla Y(p)$. Let $q \in M$ and let γ be any curve from p to q. Set

$$Z^\beta := X^\beta - Y^\beta \,, \qquad A_{\alpha\beta} = \nabla_\alpha (X_\beta - Y_\beta) \,.$$

Along the curve γ we have

$$\begin{aligned} \frac{DZ_\alpha}{ds} &= \dot\gamma^\mu \nabla_\mu Z_\alpha = \dot\gamma^\mu A_{\mu\alpha} \\ \frac{DA_{\alpha\beta}}{ds} &= \dot\gamma^\mu \nabla_\mu \nabla_\alpha Z_\beta = R_{\gamma\mu\alpha\beta} \dot\gamma^\mu Z^\gamma \,. \end{aligned} \tag{A.21.11}$$

This is a linear first order system of ODEs along γ with vanishing Cauchy data at p. Hence the solution vanishes along γ, and thus $X^\mu(q) = Y^\mu(q)$. □

Note that there are at most n values of X at p and, in view of anti-symmetry, at most $n(n-1)/2$ values of ∇X at p. Since the dimension of the space of Killing vectors equals the dimension of the group of isometries, as a corollary we obtain:

PROPOSITION A.21.8 *The dimension of the group of isometries of an n-dimensional pseudo-Riemannian manifold (M, g) is less than or equal to $n(n+1)/2$.* □

A.21.2 Analyticity of isometries

We give the proof of an unpublished result of Nomizu on analyticity of maps preserving analytic affine connections. The analyticity of isometries of analytic manifolds is a direct consequence of this:

THEOREM A.21.9 *Suppose M and M' are real analytic manifolds each provided with a real analytic affine connection. Then a (smooth) diffeomorphism f from M onto M' which preserves the affine connections is real analytic.*

PROOF: It follows from the analytic implicit function theorem that for M with an analytic affine connection ∇, the exponential mapping $\exp_p : T_pM \to M$ is real analytic on a neighborhood of 0 in T_pM, say U (so that $\exp_p(U)$ is a normal neighborhood of p). We have for each $p \in M$

$$f(\exp_p(X)) = \exp_{f(p)}(f_*(p)X) \quad \text{for every } X \in V \ ,$$

where V is an open neighborhood of 0 in T_pM. The analyticity of f follows immediately from this equation. □

A.21.3 The structure of isometry groups of asymptotically flat spacetimes

A prerequisite for studying stationary spacetimes is the understanding of the structure of the isometry groups which can arise, together with their actions. A reasonable restriction which one may wish to impose, in addition to asymptotic flatness, is that of timelikeness of the ADM momentum of the spacetimes under consideration

For the theorem that follows we do not assume anything about the nature of the Killing vectors or of the matter fields present; it is therefore convenient to use a notion of asymptotic flatness which uses at the outset four–dimensional coordinates. A metric on a set Ω will be said to be *asymptotically flat* if there exist $\alpha > 0$ and $k \geq 0$ such that

$$|g_{\mu\nu} - \eta_{\mu\nu}| + r|\partial_\alpha g_{\mu\nu}| + \cdots + r^k|\partial_{\alpha_1} \cdots \partial_{\alpha_k} g_{\mu\nu}| \leq Cr^{-\alpha} \qquad \text{(A.21.12)}$$

for some constant C ($\eta_{\mu\nu}$ is the Minkowski metric). Ω will be called *a boost–type domain*, if

$$\Omega = \{(t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3 : |\vec{x}| \geq R, |t| \leq \theta r + C\}\,, \qquad \text{(A.21.13)}$$

for some constants $\theta > 0$ and $C \in \mathbb{R}$. Let ϕ_t denote the flow of a Killing vector field X, cf. (A.21.4), p. 306. $(M, g_{\mu\nu})$ will be said to be *stationary–rotating* if the matrix of partial derivatives of X^μ asymptotically approaches a rotation matrix when $|\vec{x}|$ tends to infinity, and if ϕ_t moreover satisfies

$$\phi_{2\pi}(x^\mu) = x^\mu + A^\mu + O(r^{-\epsilon}), \qquad \epsilon > 0$$

in the asymptotically flat end, where A^μ is a timelike vector of Minkowski spacetime (in particular $A^\mu \neq 0$). One can think of $\partial/\partial\phi + a\partial/\partial t$, $a \neq 0$ as a model for the behavior involved. The interest of that definition stems from the following result, proved in [18]:

THEOREM A.21.10 (P.C. & R. Beig) *Let $(M, g_{\mu\nu})$ be a spacetime containing an asymptotically flat boost–type domain Ω, with time–like (non–vanishing) ADM four momentum p^μ, with fall–off exponent $\alpha > 1/2$ and differentiability index $k \geq 3$ (see eq. (A.21.12)). We shall also assume that the hypersurface $\{t = 0\} \subset \Omega$ can be Lorentz transformed to a hypersurface in Ω which is asymptotically orthogonal to p^μ. Suppose moreover that the Einstein tensor $G_{\mu\nu}$ of $g_{\mu\nu}$ satisfies in Ω the fall–off condition*

$$G_{\mu\nu} = O(r^{-3-\epsilon}), \qquad \epsilon > 0 \ . \tag{A.21.14}$$

Let G_0 denote the connected component of the group of all isometries of $(M, g_{\mu\nu})$. If G_0 is non–trivial, then one of the following holds:

1. *$G_0 = \mathbb{R}$, and $(M, g_{\mu\nu})$ is either stationary, or stationary–rotating.*
2. *$G_0 = U(1)$, and $(M, g_{\mu\nu})$ is axisymmetric.*
3. *$G_0 = \mathbb{R} \times U(1)$, and $(M, g_{\mu\nu})$ is stationary–axisymmetric.*
4. *$G_0 = SO(3)$, and $(M, g_{\mu\nu})$ is spherically symmetric.*
5. *$G_0 = \mathbb{R} \times SO(3)$, and $(M, g_{\mu\nu})$ is stationary–spherically symmetric.*

The reader should notice that Theorem A.21.10 excludes boost–type Killing vectors (as well as various other behavior). This feature is specific to asymptotic flatness at spatial infinity; see [31] for a large class of vacuum spacetimes with boost symmetries which are asymptotically flat in light–like directions. The theorem is sharp, in the sense that the result is not true if p^μ is allowed to vanish or to be non–time–like.

We find it likely that there exist no electro–vacuum, asymptotically flat spacetimes which have no black hole region, which are stationary–rotating and for which $G_0 = \mathbb{R}$. Some partial results concerning this can be found in [7, 26] A similar statement should be true for domains of outer communications of regular black hole spacetimes. It would be of interest to settle this question. Let us point out that the Jacobi ellipsoids [50] provide a Newtonian example of solutions with a one dimensional group of symmetries with a "stationary–rotating" behavior.

Theorem A.21.10 is used in the proof of Theorem A.21.12 below.

A.21.4 Killing vectors vs. isometry groups

In general relativity there exist at least two ways for a solution to be symmetric: there might exist

1. a Killing vector field X on the spacetime (M, g), or there might exist

2. an action of a (non–trivial) connected Lie group G on M by isometries.

Clearly 2 implies 1, *but 1 does not need to imply 2* (remove *e.g.* points from a spacetime on which an action of G exists).

Furthermore, there could also exist locally defined Killing vectors that do not extend to globally defined ones. A particularly striking example of manifolds where these notions are completely distinct is given by the flat compact manifolds constructed in [273]. In this last work Waldmüller shows that there exists a compact quotient of $\mathbb{R}^{141}$ which provides a flat manifold without any symmetries. So, while every flat n-dimensional manifold has $n(n+1)/2$ locally defined Killing vector fields, in Waldmüller's example none of them gives rise to a globally defined one. Moreover, the resulting manifold has no discrete isometries either, a fact which is usually much more difficult to establish than non-existence of continuous families of isometries.

In the uniqueness theory of stationary black holes, as presented *e.g.* in [46, 147, 150, 275] (compare [71]), one *always* assumes that an action of a group G on M exists. This is equivalent to the statement, that the orbits of all the (relevant) Killing vector fields are complete. In [60] and [61] completeness of orbits of Killing vectors was shown for vacuum and electro–vacuum spacetimes, under various conditions. The results obtained there are not completely satisfactory in the black hole context, as they do not cover degenerate black holes. Moreover, in the case of non–degenerate black holes, the theorems proved there assume that all the horizons contain their bifurcation surfaces, a condition which one may wish not to impose *a priori* in some situations.

Before stating a result which takes care of those problems, some terminology will be needed. Let Ω be a boost-type domain as defined in (A.21.12) and let $\mathscr{S}_{\rm ext}$ be the slice $\{t=0\}$ in Ω. Define the *domain of outer communications* $\langle\langle M_{\rm ext}\rangle\rangle$ as the intersection of the past and the future of the union of the orbits of the Killing vector X passing though $\mathscr{S}_{\rm ext}$:

$$M_{\rm ext} := \cup_t \phi_t(\mathscr{S}_{\rm ext})\,, \quad \langle\langle M_{\rm ext}\rangle\rangle := J^-(M_{\rm ext}) \cap J^+(M_{\rm ext})\,. \tag{A.21.15}$$

The following result, which does *not* assume any field equations, has been proved in [63]:

THEOREM A.21.11 *Consider a spacetime (M, g_{ab}) with a Killing vector field X and suppose that M contains an asymptotically flat three–end $\mathscr{S}_{\rm ext}$, with X time–like in $\mathscr{S}_{\rm ext}$. (Here the metric is assumed to be twice differentiable, while asymptotic flatness is defined in the sense of eq. (A.21.12) with $\alpha > 0$ and $k \geq 0$.) Suppose that the orbits of X are complete through all points $p \in \mathscr{S}_{\rm ext}$. If $\langle\langle M_{\rm ext}\rangle\rangle$ is globally hyperbolic, then the orbits of X through points $p \in \langle\langle M_{\rm ext}\rangle\rangle$ are complete.*

In [63] a generalization of this result to stationary–rotating spacetimes has also been given. Nomizu's theorem A.21.9 together with Theorem A.21.10 give the following result [63]:

THEOREM A.21.12 *Consider an analytic spacetime (M, g_{ab}) with a Killing vector field X with complete orbits. Suppose that M contains an asymptotically flat three–end $\mathscr{S}_{\rm ext}$ with time–like ADM four–momentum, and with $X(p)$ time–like for $p \in \mathscr{S}_{\rm ext}$. (Here asymptotic flatness is defined in the sense of eq. (A.21.12) with $\alpha > 1/2$ and $k \geq 3$, together with eq. (A.21.14) .) Let $\langle\langle M_{ext}\rangle\rangle$ denote the domain of outer communications associated with $\mathscr{S}_{\rm ext}$, and assume that $\langle\langle M_{\rm ext}\rangle\rangle$ is globally hyperbolic and simply connected. If there exists a Killing vector field Y, which is not a constant multiple of X, defined on an open subset $\mathcal{O}$ of $\langle\langle M_{\rm ext}\rangle\rangle$, then the isometry group of $\langle\langle M_{\rm ext}\rangle\rangle$ (with the metric obtained from (M, g_{ab}) by restriction) contains $\mathbb{R} \times U(1)$.*

We emphasize that no field equations or energy inequalities are assumed above. Note that simple connectedness of the domain of outer communications necessarily holds when a positivity condition is imposed on the Einstein tensor of g_{ab} [126]. Similarly the hypothesis of time–likeness of the ADM momentum will follow if one assumes existence of an appropriate space–like surface in (M, g_{ab}). It should be emphasized that no claims about isometries of $M \setminus \langle\langle M_{\rm ext}\rangle\rangle$ are made.

A.22 Null hyperplanes

One of the objects that occur in Lorentzian geometry and which posses rather disturbing properties are *null hyperplanes* and *null hypersurfaces*, and it appears useful to include a short discussion of those. Perhaps the most unusual feature of such objects is that the direction normal is actually tangential as well. Furthermore, because the normal has no natural normalization, there is no natural measure induced on a null hypersurface by the ambient metric.

In this section we present some algebraic preliminaries concerning null hyperplanes, null hypersurfaces will be discussed in Section A.23 below.

Let W be a real vector space, and recall that its dual W^* is defined as the set of all linear maps from W to $\mathbb{R}$ in the applications (in this work only vector spaces over the reals are relevant, but the field makes no difference for the discussion below). To avoid unnecessary complications we assume that W is finite dimensional. It is then standard that W^* has the same dimension as W.

We suppose that W is equipped with a *a) bilinear, b) symmetric, and c) non-degenerate form q.* Thus

$$q : W \times W \to \mathbb{R}$$

satisfies

$$a) \quad q(\lambda X + \mu Y, Z) = \lambda q(X, Z) + \mu q(Y, Z)\,, \qquad b) \quad q(X, Y) = q(Y, X)\,,$$

and we also have the implication

$$c) \quad \forall Y \in W \; q(X, Y) = 0 \implies X = 0\,. \qquad \text{(A.22.1)}$$

(Strictly speaking, we should have indicated linearity with respect to the second variable in a) as well, but this property follows from a) and b) as above). By an abuse of terminology, we will call q a *scalar product*; note that standard algebra textbooks often add the condition of positive-definiteness to the definition of scalar product, which we do not include here.

Let $V \subset W$ be a vector subspace of W. The *annihilator* V^0 of W is defined as the set of linear forms on W which vanish on V:

$$V^0 := \{\alpha \in W^* : \ \forall Y \in V \quad \alpha(Y) = 0\} \subset W^* .$$

V^0 is obviously a linear subspace of W^*.

Because q non-degenerate, it defines a linear isomorphism, denoted by $\flat$, between W and W^* by the formula:

$$X^\flat(Y) = q(X, Y) .$$

Indeed, the map $X \mapsto X^\flat$ is clearly linear. Next, it has no kernel by (A.22.1). Since the dimensions of W and W^* are the same, it must be an isomorphism. The inverse map is denoted by $\sharp$. Thus, by definition we have

$$q(\alpha^\sharp, Y) = \alpha(Y) .$$

The map $\flat$ is nothing but "the lowering of the index on a vector using the metric q", while $\sharp$ is the "raising of the index on a one-form using the inverse metric".

For further purposes it is useful to recall the standard fact:

PROPOSITION A.22.1

$$\dim V + \dim V^0 = \dim W .$$

PROOF: Let $\{e_i\}_{i=1,\ldots,\dim V}$ be any basis of V, we can complete $\{e_i\}$ to a basis $\{e_i, f_a\}$, with $a = 1, \ldots, \dim W - \dim V$, of W. Let $\{e_i^*, f_a^*\}$ be the dual basis of W^*. It is straightforward to check that V^0 is spanned by $\{f_a^*\}$, which gives the result. □

The quadratic form q defines the notion of orthogonality:

$$V^\perp := \{Y \in W : \forall X \in V \ q(X, Y) = 0\} .$$

A chase through the definitions above shows that

$$V^\perp = (V^0)^\sharp .$$

Proposition A.22.1 implies:

PROPOSITION A.22.2

$$\dim V + \dim V^\perp = \dim W .$$

This implies, again regardless of signature:

PROPOSITION A.22.3

$$(V^\perp)^\perp = V\,.$$

PROOF: The inclusion $(V^\perp)^\perp \supset V$ is obvious from the definitions. The equality follows now because both spaces have the same dimension, as a consequence of Proposition (A.22.2). □

Now,

$$X \in V \cap V^\perp \implies q(X,X) = 0\,, \tag{A.22.2}$$

so that X vanishes if q is positive- or negative-definite, leading to $\dim V \cap \dim V^\perp = \{0\}$ in those cases. However, this does not have to be the case anymore for non-definite scalar products q.

A vector subspace V of W is called a *hyperplane* if

$$\dim V = \dim W - 1\,.$$

Proposition A.22.2 implies then

$$\dim V^\perp = 1\,,$$

regardless of the signature of q. Thus, given a hyperplane V there exists a vector w such that

$$V^\perp = \mathbb{R}w\,.$$

If q is Lorentzian, we say that

$$V \text{ is } \begin{cases} \text{spacelike} & \text{if } w \text{ is timelike;} \\ \text{timelike} & \text{if } w \text{ is spacelike;} \\ \text{null} & \text{if } w \text{ is null.} \end{cases}$$

An argument based e.g. on Gram-Schmidt orthonormalization shows that if V is spacelike, then the scalar product defined on V by restriction is positive-definite; similarly if V is timelike, then the resulting scalar product is Lorentzian. The last case, of a null V, leads to a degenerate induced scalar product. In fact, we claim that

$$V \text{ is null if and only if } V \text{ contains its normal.}. \tag{A.22.3}$$

To see (A.22.3), suppose that $V^\perp = \mathbb{R}w$, with w null. Since $q(w,w) = 0$ we have $w \in (\mathbb{R}w)^\perp$, and from Proposition A.22.3

$$w \in (\mathbb{R}w)^\perp = (V^\perp)^\perp = V\,.$$

Since V does not contain its normal in the remaining cases, the equivalence is established.

As discussed in more detail in the next section, a hypersurface $\mathscr{N} \subset \mathscr{M}$ is called *null* if at every $p \in \mathscr{N}$ the scalar product restricted to $T_p\mathscr{N}$ is degenerate. Equivalently, the tangent space $T_p\mathscr{N}$ is a null subspace of $T_p\mathscr{M}$. So (A.22.2) shows that vectors normal to a null hypersurface $\mathscr{N}$ are also tangent to $\mathscr{N}$.

A.23 The geometry of null hypersurfaces

In this section we review some aspects of the geometry of null hypersurfaces We follow the exposition in [73], which in turn is based on [127].

A C^k *null hypersurface* in a spacetime $(\mathscr{M}, g)$, $k \geq 1$, is a C^k co-dimension one embedded submanifold $\mathscr{N}$ of $\mathscr{M}$ such that the pullback of the metric g to $\mathscr{N}$ is degenerate. Each such hypersurface $\mathscr{N}$ admits a C^{k-1} non-vanishing *future directed* null vector field $L \in \Gamma T\mathscr{N}$ such that the normal space of L at a point $p \in \mathscr{N}$ coincides with the tangent space of $\mathscr{N}$ at p, *i.e.*, $L_p^\perp = T_p\mathscr{N}$ for all $p \in \mathscr{N}$. In particular, tangent vectors to $\mathscr{N}$ not parallel to L are spacelike. We note that the vector field L is unique up to a positive scale factor.

The integral curves of L, when suitably parameterized, are null geodesics, called the *null geodesic generators* of $\mathscr{N}$: Indeed, since $g(L, L) = 0$ on $\mathscr{N}$, we have that $X(g(L, L)) = 0$ for all vectors X tangent to $\mathscr{N}$. This implies that $d(g(L, L))$ annihilates $T\mathscr{N}$, hence is conormal to $\mathscr{N}$. Now, $g(L, \cdot)$ annihilates $T\mathscr{N}$ as well, so we conclude that

$$\nabla(g(L, L)) \sim L\,.$$

Let u be any defining function for $\mathscr{N}$, i.e., $\mathscr{N} = \{u = 0\}$, with du nowhere vanishing on $\mathscr{N}$. Since du annihilates $T\mathscr{N}$ as well, L must be proportional to ∇u, so let us first assume that $L = \nabla u$. Then

$$\begin{aligned} \nabla_\mu(g(L, L)) &= \nabla_\mu(L^\alpha L_\alpha) = 2L^\alpha \nabla_\mu L_\alpha = 2L^\alpha \nabla_\mu \nabla_\alpha u = 2L^\alpha \nabla_\alpha \nabla_\mu u \\ &= 2L^\alpha \nabla_\alpha L_\mu \sim L_\mu\,. \end{aligned} \tag{A.23.1}$$

We have thus shown that the integral curves of ∇u are null geodesics, though perhaps not affinely parameterized. Now, multiplying ∇u by a function will not change its integral curves, but only their parameterisation, so the result remains true for all L proportional to ∇u.

Since L is orthogonal to $\mathscr{N}$ we can introduce the *null Weingarten map* and *null second fundamental form* of $\mathscr{N}$ with respect L in a manner roughly analogous to what is done for spacelike hypersurfaces or hypersurfaces in a Riemannian manifold, as follows: We start by introducing an equivalence relation on tangent vectors: for $X, X' \in T_p\mathscr{N}$, $X' = X \bmod L$ if and only if $X' - X = \lambda L$ for some $\lambda \in \mathbb{R}$. Let $\overline{X}$ denote the equivalence class of X. Now, if $X' = X \bmod L$ and $Y' = Y \bmod L$ then obviously

$$\langle X', Y' \rangle = \langle X, Y \rangle\,, \tag{A.23.2}$$

since $g(L, Z)$ vanishes for all $Z \in T\mathscr{N}$. Similarly, writing $X' = X + aL$ for some function a,

$$\langle \nabla_{X'} L, Y' \rangle = \langle \nabla_{X+aL} L, Y' \rangle = \langle \nabla_X L + a \underbrace{\nabla_L L}_{\sim L}, Y' \rangle = \langle \nabla_X L, Y \rangle\,,$$

Hence, for various quantities of interest, components along L are irrelevant. For this reason one works with the tangent space of $\mathscr{N}$ modded out by L, *i.e.*,

$$(T_p\mathscr{N})/L = \{\overline{X} \mid X \in T_p\mathscr{N}\} \text{ and } (T\mathscr{N})/L = \cup_{p \in \mathscr{N}} (T_p\mathscr{N})/L;\,.$$

The bundle $(T\mathscr{N})/L$ is a vector bundle over $\mathscr{N}$ of dimension $(n-1)$, and does not depend on the particular choice of null vector field L.

There is a natural positive definite metric h in $(T\mathscr{N})/L$ induced from g: For each $p \in \mathscr{N}$, define $h\colon (T_p\mathscr{N})/L \times (T_p\mathscr{N})/L \to \mathbb{R}$ by

$$h(\overline{X}, \overline{Y}) = \langle X, Y\rangle\,. \tag{A.23.3}$$

Equation (A.23.2) shows that h is well-defined.

The *null Weingarten map* $b = b_L$ of $\mathscr{N}$ with respect to L is, for each point $p \in \mathscr{N}$, a linear map

$$b\colon (T_p\mathscr{N})/L \to (T_p\mathscr{N})/L$$

defined by

$$b(\overline{X}) = \overline{\nabla_X L}\,. \tag{A.23.4}$$

To see that b is well-defined, let $X' = X + \lambda L$, then

$$\nabla_{X'} L = \nabla_{X+\lambda L} L = \nabla_X L + \lambda \underbrace{\nabla_L L}_{\sim L},$$

which shows that the equivalence classes $\overline{\nabla_{X'} L}$ and $\overline{\nabla_X L}$ coincide, as needed.

If $\widetilde{L} = fL$, $f \in C^1(\mathscr{N})$, is any other future directed null vector field tangent to $\mathscr{N}$, then

$$\nabla_X \widetilde{L} = f\nabla_X L \bmod L\,.$$

Thus

$$b_{fL} = f b_L\,. \tag{A.23.5}$$

It follows that the Weingarten map b of $\mathscr{N}$ is defined *only up to a positive scale* factor.

The *null second fundamental form* $B = B_L$ of $\mathscr{N}$ with respect to L is the bilinear form associated to b via h: For each $p \in \mathscr{N}$, the map

$$B\colon (T_p\mathscr{N})/L \times (T_p\mathscr{N})/L \to \mathbb{R}$$

is defined by

$$B(\overline{X}, \overline{Y}) := h(b(\overline{X}), \overline{Y}) = \langle \nabla_X L, Y\rangle\,. \tag{A.23.6}$$

Now,

$$h(b(\overline{X}), \overline{Y}) = \langle \nabla_X L, Y\rangle = \langle X, \nabla_Y L\rangle = h(\overline{X}, b(\overline{Y}))\,. \tag{A.23.7}$$

This shows that b is self-adjoint with respect to h, and that B is symmetric.

> INCIDENTALLY: In a manner analogous to the second fundamental form for spacelike hypersurfaces, a null hypersurface is totally geodesic if and only if B vanishes identically [180, Theorem 30]. □

The *null mean curvature* of $\mathscr{N}$ with respect to L is the continuous scalar field $\theta \in C^0(\mathscr{N})$ defined by

$$\theta = \mathrm{tr}_h\, b\,; \tag{A.23.8}$$

called *divergence*, or *expansion*, of the horizon. Let $e_1, e_2, \ldots, e_{n-1}$ be $n-1$ orthonormal spacelike vectors (with respect to g) tangent to $\mathscr{N}$ at p. Then $\{\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_{n-1}\}$ is an orthonormal basis (with respect to h) of $(T_p\mathscr{N})/L$. Hence at p,

$$\begin{aligned}\theta &= \operatorname{tr} b = \textstyle\sum_{i=1}^{n-1} h(b(\overline{e}_i), \overline{e}_i) \\ &= \textstyle\sum_{i=1}^{n-1} \langle \nabla_{e_i} L, e_i \rangle . \end{aligned} \tag{A.23.9}$$

Let Σ be the intersection, transverse to L, of a hypersurface in $\mathscr{M}$ with $\mathscr{N}$. Then Σ is a C^2 $(n-1)$–dimensional spacelike submanifold of $\mathscr{M}$ contained in $\mathscr{N}$ which meets L orthogonally. From (A.23.9), by definition of the divergence along a submanifold,

$$\theta|_\Sigma = \mathrm{div}_\Sigma L \,,$$

and hence the null mean curvature gives a measure of the divergence of the null generators of $\mathscr{N}$.

INCIDENTALLY: Given Σ and L as above, let $\underline{L}$ be the second future directed null vector normal to Σ normalised so that

$$g(\underline{L}, L) = -2 \,. \tag{A.23.10}$$

Set

$$h_{\mu\nu} := g_{\mu\nu} + \frac{1}{2}(\underline{L}_\mu L_\nu + \underline{L}_\nu L_\mu) \,. \tag{A.23.11}$$

Then $h^\mu{}_\nu := g^{\mu\alpha} h_{\alpha\nu}$ is the projector on $T\Sigma$: indeed, $X \in T\Sigma$ if and only if $g(X, \underline{L}) = g(X, L) = 0$, but then $h^\mu{}_\nu X^\nu = X^\mu$ readily follows. On the other hand, $h^\mu{}_\nu L^\nu = h^\mu{}_\nu \underline{L}^\nu = 0$ holds by definition of h.

Along Σ, elements of $T\mathscr{N}/L$ can be represented by vectors in $T\Sigma$, then h in (A.23.11) coincides with h of (A.23.3), and the former can be thought of a spacetime equivalent of the latter. Note, however, that the definition (A.23.3) does not require any supplementary structures, while (A.23.11) requires Σ, or at least $\underline{L}$.

From the definition of θ we have

$$\begin{aligned}\theta &= h^{\mu\nu}\nabla_\mu L_\nu \\ &= g^{\mu\nu}\nabla_\mu L_\nu + \frac{1}{2}(\underline{L}^\mu L^\nu \nabla_\mu L_\nu + \underline{L}^\nu L^\mu \nabla_\mu L_\nu) \,. \end{aligned} \tag{A.23.12}$$

In some situations L arises from a null vector field defined in a neighborhood of $\mathscr{N}$, and is autoparallel on $\mathscr{N}$. Then $g(L, L) = 0$ near $\mathscr{N}$, which implies that the middle term in (A.23.12) drops out. Further, $\nabla_L L = 0$ on $\mathscr{N}$, which shows that the last terms drops out. One then finds a convenient formula:

$$\text{If } g(L,L) = 0 \text{ near } \mathscr{N} \text{ and if } \nabla_L L = 0 \text{ on } \mathscr{N} \text{ then } \theta = \nabla_\mu L^\mu \,. \tag{A.23.13}$$

(This should be compared with the formula $H = \nabla_\mu n^\mu$ for the mean curvature H of a non-characteristic hypersurface $\mathscr{S}$, where n is a unit normal to $\mathscr{S}$, without the need of any further hypotheses.)

EXAMPLE A.23.3 Let $\mathscr{N}$ be a null hypersurface $u := x - t = 0$ in Minkowski spacetime; we choose $-t+x$ instead of $-x+t$ because of the convention that $L := \nabla u$ in the calculations above should be future pointing. Then L is null everywhere, and autoparallel everywhere, thus

$$\theta = \nabla_\alpha \nabla^\alpha u = \Box_g u = 0 \,.$$

Next, still in Minkowski spacetime, let $\mathscr{N}$ be a future null cone $\mathscr{N} = \dot{I}^+(p) - \{p\}$, $u := -t + r = 0$. Then ∇u is null, geodesic, and future pointing, so

$$\theta = \Box u = \partial_\mu(\eta^{\mu\nu}\partial_\mu u) = \partial_\mu(\eta^{\mu\nu}\partial_\mu r) = \partial_i(\eta^{ij}\partial_j r) = \partial_i\left(\frac{x^i}{r}\right) = \frac{n-1}{r}\,. \quad \text{(A.23.14)}$$

Hence $\dot{I}^+(p) - \{p\}$ has positive null mean curvature. Changing time-orientation, we obtain that a past null cone $\mathscr{N} = \dot{I}^-(p) - \{p\}$ in Minkowski spacetime has negative null mean curvature, $\theta < 0$.

Finally, in an arbitrary smooth spacetime, consider a future null cone $\mathscr{N} = \dot{I}^+(p) - \{p\}$ near its tip. In geodesic coordinates $\dot{I}^+(p)$ is given again by the equation $-t + r = 0$. In those coordinates the metric coincides with the Minkowskian metric up to terms quadratic in the coordinates, and the derivatives of the metric vanish at p. With a little work one checks that (A.23.14) holds near p up to terms which are $O(1)$.

Example A.23.4 Let $\mathscr{S}$ be a spacelike hypersurface in $\mathscr{N}$ with induced metric γ and extrinsic curvature K, and let Σ be the intersection of $\mathscr{N}$ with $\mathscr{S}$; this is a smooth submanifold of $\mathscr{S}$, of codimension one in $\mathscr{S}$, since the intersection is transverse. Let n be the field of unit normals to Σ within $\mathscr{S}$, we can chose both L and the direction of n so that

$$L = T + n\,,$$

where T is the field of unit normals to $\mathscr{S}$ in $\mathscr{M}$. Let h denote the metric induced by g on Σ. Representing, as before, elements of $T\mathscr{M}/L$ by vectors tangents to Σ, one finds, in local coordinates x^A on $\mathscr{S}$,

$$b_{AB} = K_{AB} + \lambda_{AB}\,, \qquad \theta = h^{AB}K_{AB} + H\,, \quad \text{(A.23.15)}$$

where λ_{AB} is the extrinsic curvature (second fundamental form) of Σ within $(\mathscr{S}, \gamma)$, and H is the mean curvature of Σ within $(\mathscr{S}, \gamma)$.

□

Equation (A.23.5) shows that if $\widetilde{L} = fL$, then $\widetilde{\theta} = f\theta$. Thus the null mean curvature inequalities $\theta \geq 0$, $\theta \leq 0$, are invariant under positive rescaling of L.

A.24 Elements of causality theory

We collect here some definitions from causality theory. Given a manifold $\mathscr{M}$ equipped with a Lorentzian metric g, at each point $p \in \mathscr{M}$ the set of timelike vectors in T_pM has precisely two components. A *time-orientation of* $T_p\mathscr{M}$ is the assignment of the name "future pointing vectors" to one of those components; vectors in the remaining component are then called "past pointing". A Lorentzian manifold is said to be *time-orientable* if such locally defined time-orientations can be defined globally in a consistent way. A *spacetime* is a time-orientable Lorentzian manifold on which a time-orientation has been chosen.

A differentiable path γ will be said to be *timelike* if at each point the tangent vector $\dot{\gamma}$ is timelike; it will be said *future directed* if $\dot{\gamma}$ is future directed. There is an obvious extension of this definition to *null, causal* or *spacelike* curves. We define an *observer* to be an *inextendible, future directed timelike path.* In these notes the names "path" and "curve" will be used interchangeably.

Let $\mathscr{U} \subset \mathscr{O} \subset \mathscr{M}$. One sets

$$\begin{aligned} I^+(\mathscr{U};\mathscr{O}) &:= \{q \in \mathscr{O} : \text{there exists a timelike future directed path} \\ &\qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\}\,, \\ J^+(\mathscr{U};\mathscr{O}) &:= \{q \in \mathscr{O} : \text{there exists a causal future directed path} \\ &\qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\} \cup \mathscr{U}\,. \end{aligned}$$

$I^-(\mathscr{U};\mathscr{O})$ and $J^-(\mathscr{U};\mathscr{O})$ are defined by replacing "future" by "past" in the definitions above. The set $I^+(\mathscr{U};\mathscr{O})$ is called the *timelike future of* $\mathscr{U}$ *in* $\mathscr{O}$, while $J^+(\mathscr{U};\mathscr{O})$ is called the *causal future of* $\mathscr{U}$ *in* $\mathscr{O}$, with similar terminology for the timelike past and the causal past. We will write $I^\pm(\mathscr{U})$ for $I^\pm(\mathscr{U};\mathscr{M})$, similarly for $J^\pm(\mathscr{U})$, and one then omits the qualification "in $\mathscr{M}$" when talking about the causal or timelike futures and pasts of $\mathscr{U}$. We will write $I^\pm(p;\mathscr{O})$ for $I^\pm(\{p\};\mathscr{O})$, $I^\pm(p)$ for $I^\pm(\{p\};\mathscr{M})$, *etc.*

A function f will be called a *time function* if its gradient is *timelike, past pointing*. Similarly a function f will be said to be a *causal function* if its gradient is *causal, past pointing*. The choice "past-pointing" here has to do with our choice $(-,+,\dots,+)$ of the signature of the metric. This is easily understood on the example of Minkowski spacetime $(\mathbb{R}^{n+1},\eta)$, where the gradient of the usual time coordinate t is $-\partial_t$, since $\eta^{00}=-1$. Had we chosen to work with the signature $(+,-,\dots,-)$, time functions would have been defined to have future pointing gradients.

A differentiable hypersurface $\mathscr{S} \subset \mathscr{M}$ is called a *Cauchy surface* if every inextendible causal curve intersects $\mathscr{S}$ precisely once. A spacetime is called *globally hyperbolic* if it contains a Cauchy hypersurface. A key property of globally hyperbolic spacetimes is, that they possess a time-function t (in fact, many) with the property that each level set of t is a Cauchy surface.

A spacetime $(\mathscr{M},g)$ is called *maximal globally hyperbolic* if it is globally hyperbolic and if there exists no spacetime $(\widetilde{\mathscr{M}},\widetilde{g})$ such that $(\mathscr{M},g)$ is a proper subset of $(\widetilde{\mathscr{M}},\widetilde{g})$.

The reader is referred to [67, 76, 115, 142, 178, 179, 205, 267] for extensive modern treatments of causality theory, including applications to incompleteness theorems (also known as "singularity theorems").

Appendix B

A collection of identities

We include here a collection of useful identities, mostly compiled by Erwann Delay. I am grateful to Erwann for allowing me to include his list here.

B.1 ADM notation

Letting $\tilde g^{ij}$ denote the inverse matrix to g_{ij}, using the Arnowitt-Deser-Misner notation we have

$$g^{kl} = \tilde g^{kl} - \tfrac{N^l N^k}{N^2},\ g_{0k} = N_k,\ g^{0k} = \tfrac{N^k}{N^2},\ N^2 = -\tfrac{1}{g^{00}},\ g_{00} = N^k N_k - N^2. \tag{B.1.1}$$

where $N^k := \tilde g^{kl} N_l$. The associated decomposition of the Christoffel symbols reads

$$\Gamma^0_{k0} = \partial_k \log N - \frac{N^l}{N} K_{lk}\,, \quad \Gamma^0_{00} = \partial_0 \log N + N^k \partial_k \log N - \frac{N^l N^k}{N} K_{lk}$$

(recall that $K_{kl} = -N\Gamma^0_{kl} = \frac{1}{2N}(D_l N_k + D_k N_l - \partial_0 g_{kl})$). Furthermore,

$$\Gamma^k_{ij} = \tilde\Gamma^k_{ij} + \frac{N^k}{N} K_{ij}\,, \quad \Gamma^k_{0j} = D_j N^k - N K^k{}_j + \frac{N^k}{N}\Big(N^l K_{lj} - D_j N\Big)\,.$$

B.2 Some commutators

Here are some formulae for the commutation of derivatives:

$$\nabla_m \nabla_l t_{ik} - \nabla_l \nabla_m t_{ik} = R^p{}_{klm} t_{ip} + R^p{}_{ilm} t_{kp}\,,$$

$$\nabla_i \nabla_j V^l - \nabla_j \nabla_i V^l = R^l{}_{kij} V^k\,,$$

$$\nabla^k \nabla_k |df|^2 = 2(\nabla^l f \nabla_l \nabla^k \nabla_k f + Ric(\nabla f, \nabla f) + |\nabla\nabla f|^2)\,,$$

$$\nabla^k \nabla_k \nabla_i \nabla_j f - \nabla_i \nabla_j \nabla^k \nabla_k f - R_{kj} \nabla^k \nabla_i f - R_{ki} \nabla^k \nabla_j f + 2R_{qjli} \nabla^q \nabla^l f$$
$$= (\nabla_i R_{kj} + \nabla_j R_{ki} - \nabla_k R_{ij}) \nabla^k f\,.$$

B.3 Bianchi identities

The Bianchi identities for a Levi-Civita connection:

$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} = 0\,,$$
$$\nabla_l R^t{}_{ijk} + \nabla_k R^t{}_{ilj} + \nabla_j R^t{}_{ikl} = 0\,,$$
$$\nabla_t R^t{}_{ijk} + \nabla_k R_{ij} - \nabla_j R_{ik} = 0\,,$$
$$\nabla^k R_{ik} - \frac{1}{2}\nabla_k R = 0\,.$$

B.4 Linearisations

Linearisations for various objects of interest:

$$D_g\Gamma^k_{ij}(g)h = \frac{1}{2}(\nabla_i h^k_j + \nabla_j h^k_i - \nabla^k h_{ij})\,,$$
$$2[D_g\mathrm{Riem}(g)h]_{sklm} = \nabla_l\nabla_k h_{sm} - \nabla_l\nabla_s h_{km} + \nabla_m\nabla_s h_{kl} - \nabla_m\nabla_k h_{sl} + R^p{}_{klm}h_{ps} + R^p{}_{sml}h_{pk}\,,$$
$$2[D_g\mathrm{Riem}(g)h]^i_{klm} = \nabla_l\nabla_k h^i_m - \nabla_l\nabla^i h_{km} + \nabla_m\nabla^i h_{kl} - \nabla_m\nabla_k h^i_l + g^{is}R^p{}_{sml}h_{pk} - R^p{}_{klm}h^i_p\,,$$
$$D_g Ric(g)h = \frac{1}{2}\Delta_L h - \mathrm{div}^*\mathrm{div}(Gh)\,,$$
$$\Delta_L h_{ij} = -\nabla^k\nabla_k h_{ij} + R_{ik}h^k{}_j + R_{jk}h^k{}_j - 2R_{ikjl}h^{kl}\,,$$
$$Gh = h - \frac{1}{2}tr\, hg, \quad (\mathrm{div}h)_i = -\nabla^k h_{ik}, \quad \mathrm{div}^* w = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i)\,,$$
$$D_g R(g)h = -\nabla^k\nabla_k(tr\ h) + \nabla^k\nabla^l h_{kl} - R^{kl}h_{kl}\,,$$
$$[D_g R(g)]^* f = -\nabla^k\nabla_k f g + \nabla\nabla f - f\ Ric(g)\,.$$

B.5 Warped products

Let (M, g), $\nabla := \nabla_g$, $f : M \to \mathbb{R}$ and

$$(\mathcal{M} = M \times_f I, \widetilde{g} = -f^2 dt^2 + g)\,,$$

then for X, Y tangent to M and V, W tangent to I, we have

$$Ric(\widetilde{g})(X, Y) = Ric(g)(X, Y) - f^{-1}\nabla\nabla f(X, Y)\,,$$
$$Ric(\widetilde{g})(X, V) = 0 = \widetilde{g}(X, V)\,,$$
$$Ric(\widetilde{g})(V, W) = -f^{-1}\nabla^k\nabla_k f\widetilde{g}(V, W)\,.$$

Let (M, g), $\nabla := \nabla_g$, $f : M \to \mathbb{R}$ and let $(\mathcal{M} = M \times_f I, \widetilde{g} = \epsilon f^2 dt^2 + g)$, $\epsilon = \pm 1$. $x^a = (x^0 = t, x^i = (x^1, \ldots, x^n))$.

$$\widetilde{\Gamma}^0_{00} = \widetilde{\Gamma}^0_{ij} = \widetilde{\Gamma}^k_{i0} = 0, \ \widetilde{\Gamma}^0_{i0} = f^{-1}\partial_i f, \ \widetilde{\Gamma}^k_{00} = -\epsilon f\nabla^k f, \ \widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij}\,,$$
$$\widetilde{R}^l{}_{ijk} = R^l{}_{ijk}, \ \widetilde{R}^l{}_{0j0} = -\epsilon f\nabla_j\nabla^l f, \ \widetilde{R}^0{}_{ij0} = f^{-1}\nabla_j\nabla_i f\,,$$
$$\widetilde{R}^0{}_{ijk} = \widetilde{R}^l{}_{ij0} = \widetilde{R}^l{}_{0jk} = \widetilde{R}^0{}_{0jk} = \widetilde{R}^0{}_{0j0} = 0\,,$$
$$\widetilde{R}_{mijk} = R_{mijk}, \ \widetilde{R}_{0ijk} = 0, \ \widetilde{R}_{0ij0} = \epsilon f\nabla_j\nabla_i f\,,$$
$$\widetilde{R}_{ik} = R_{ik} - f^{-1}\nabla_k\nabla_i f, \ \widetilde{R}_{0k} = 0, \ \widetilde{R}_{00} = -\epsilon f\nabla^i\nabla_i f\,,$$
$$\widetilde{R} = R - 2f^{-1}\nabla^i\nabla_i f\,.$$

B.6 Hypersurfaces

Let M be a non-isotropic hypersurface in $\widetilde{M}$, with ν normal, and u, v tangent to M at m, we have

$$II(u,v) = (\widetilde{\nabla}_U V - \nabla_U V)_m = (\widetilde{\nabla}_U V)_m^{\perp} = II(v,u) = -l(u,v)\nu_m \,.$$

Setting $S(u) = \widetilde{\nabla}_u \nu \in T_m M$, one has

$$< S(u), v >=< \widetilde{\nabla}_u \nu, v >=< -\nu, \widetilde{\nabla}_u V >= l(u,v) \,.$$

If x, y, u, v are tangent to M, then

$$R(x,y,u,v) = \widetilde{R}(x,y,u,v) + l(x,u)l(y,v) - l(x,v)l(y,u) \,.$$

The *Gauss-Codazzi* equations read

$$\widetilde{R}(x,y,u,\nu) = \nabla_y l(x,u) - \nabla_x l(y,u) \,.$$

The Ricci tensor can be decomposed as:

$$\widetilde{R}(y,v) = R(y,v) + II \circ II(y,v) - trII \; II(y,v) + \widetilde{R}(\nu, y, \nu, v) \,,$$

$$\widetilde{R}(y,\nu) = -\nabla_y trII + y^j \nabla^i II_{ij} \,,$$

$$\widetilde{R} = R + |II|^2 - (trII)^2 + 2\widetilde{R}(\nu,\nu) \,.$$

B.7 Conformal transformations

The Weyl tensor:

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}) \,.$$

We have

$$W_i{}^j{}_{kl}(e^f g) = W_i{}^j{}_{kl}(g) \,.$$

The Schouten tensor

$$S_{ij} = \frac{1}{n-2}[2R_{ij} - \frac{R}{n-1}g_{ij}] \,.$$

Under a conformal transformation $g' = e^f g$, we have

$$\Gamma'^k_{ij} - \Gamma^k_{ij} = \frac{1}{2}(\delta^k_j \partial_i f + \delta^k_i \partial_j f - g_{ij}\nabla^k f) \,.$$

$$R'_{ij} = R_{ij} - \frac{n-2}{2}\nabla_i\nabla_j f + \frac{n-2}{4}\nabla_i f \nabla_j f - \frac{1}{2}(\nabla^k\nabla_k f + \frac{n-2}{2}|df|^2)g_{ij}$$

$$R' = e^{-f}[R - (n-1)\nabla^i\nabla_i f - \frac{(n-1)(n-2)}{4}\nabla^i f \nabla_i f] \,.$$

Specialising to $g' = e^{\frac{2}{n-2}u} g$,

$$R'_{ij} = R_{ij} - \nabla_i \nabla_j u + \frac{1}{n-2} \nabla_i u \nabla_j u - \frac{1}{n-2} (\nabla^k \nabla_k u + |du|^2) g_{ij} \,.$$

In the notation $g' = v^{\frac{2}{n-2}} g$,

$$R'_{ij} = R_{ij} - v^{-1} \nabla_i \nabla_j v + \frac{n-1}{n-2} v^{-2} \nabla_i v \nabla_j v - \frac{1}{n-2} v^{-1} (\nabla^k \nabla_k v) g_{ij} \,.$$

If we write instead $g' = \phi^{4/(n-2)} g$, then

$$R'_{ij} = R_{ij} - 2\phi^{-1} \nabla_i \nabla_j \phi + \frac{2n}{n-2} \phi^{-2} \nabla_i \phi \nabla_j \phi - \frac{2}{n-2} \phi^{-1} (\nabla^k \nabla_k \phi + \phi^{-1} |d\phi|^2) g_{ij} \,,$$

$$R' \phi^{(n+2)/(n-2)} = -\frac{4(n-1)}{n-2} \nabla^k \nabla_k \phi + R\phi \,.$$

When we have two metrics g and g' at our disposal, then

$$T^k_{ij} := \Gamma'^k_{ij} - \Gamma^k_{ij} = \frac{1}{2} g'^{kl} (\nabla_i g'_{lj} + \nabla_j g'_{li} - \nabla_l g'_{ij}) \,.$$

$$\mathrm{Riem}'^i{}_{klm} - \mathrm{Riem}^i{}_{klm} = \nabla_l T^i_{km} - \nabla_m T^i_{kl} + T^i_{jl} T^j_{km} - T^i_{jm} T^j_{kl} \,.$$

Under $g' = e^f g$, the Laplacian acting on functions transforms as

$$\nabla'^k \nabla'_k v = e^{-f} (\nabla^k \nabla_k v + \frac{n-2}{2} \nabla^k f \nabla_k v) \,.$$

For symmetric tensors we have instead

$$\begin{aligned} \nabla'^k \nabla'_k u_{ij} \;=\; e^{-f} \Big[& \nabla^k \nabla_k u_{ij} + \frac{n-6}{2} \nabla^k f \nabla_k u_{ij} - (\nabla_i f \nabla^k u_{kj} + \nabla_j f \nabla^k u_{ki}) \\ & + (\nabla^k f \nabla_i u_{kj} + \nabla^k f \nabla_j u_{ki}) + (\frac{3-n}{2} \nabla^k f \nabla_k f - \nabla^k \nabla_k f) u_{ij} \\ & - \frac{n}{4} (\nabla_i f \nabla^k f u_{kj} + \nabla_j f \nabla^k f u_{ki}) + \frac{1}{2} \nabla_i f \nabla_j f u^k_k + \frac{1}{2} g_{ij} u_{kl} \nabla^k f \nabla^l f \Big] \,. \end{aligned}$$

B.8 Laplacians on tensors

For symmetric u's and arbitrary T's let

$$(Du)_{kij} := \frac{1}{\sqrt{2}} (\nabla_k u_{ij} - \nabla_j u_{ik}),$$

then

$$(D^* T)_{ij} = \frac{1}{2\sqrt{2}} (-\nabla^k T_{kij} - \nabla^k T_{kji} + \nabla^k T_{ijk} + \nabla^k T_{jik}) \,.$$

Further

$$D^* D u_{ij} = -\nabla^k \nabla_k u_{ij} + \frac{1}{2} (\nabla^k \nabla_i u_{jk} + \nabla^k \nabla_j u_{ik}) \,,$$

and

$$\mathrm{div}^* \mathrm{div} u = -\frac{1}{2} (\nabla_i \nabla^k u_{jk} + \nabla_j \nabla^k u_{ik}) \,,$$

thus

$$(D^* D + \mathrm{div}^* \mathrm{div}) u_{ij} = -\nabla^k \nabla_k u_{ij} + \frac{1}{2} (R_{kj} u^k_i + R_{ki} u^k_j - 2R_{qjli} u^{ql}) \,.$$

B.9 Stationary metrics

Let (M, γ) be a Riemannian or pseudo-Riemannian three dimensional manifold, define $\lambda : M \to \mathbb{R}$, $\xi : M \to T^*M$, $(N = I \times M, g)$ by the formulae

$$g(t,x) = \begin{pmatrix} \lambda & {}^t\xi \\ \xi & \lambda^{-1}(\xi\ {}^t\xi - \gamma) \end{pmatrix} = \lambda(dt + \lambda^{-1}\xi_i dx^i)^2 - \lambda^{-1}\gamma_{ij}dx^i dx^j \,.$$

Let $w = -\lambda^2 *_\gamma d(\lambda^{-1}\xi)$. $\nabla = \nabla_g$, $E^i = \gamma^{is}E_s$. Then

$$\mathrm{Ric}(\gamma)_{ij} = \frac{1}{2}\lambda^{-1}(\nabla_i\lambda\nabla_j\lambda + w_i w_j) + \lambda^{-2}(Ric(g)_{ij} - Ric(g)_{cd}\xi^c\xi^d\gamma_{ij})\,,$$

$$\nabla^i\nabla_i\lambda = \lambda^{-1}(|d\lambda|^2 - |w|^2) - 2\lambda^{-1}Ric(g)_{ab}\xi^a\xi^b\,,$$

$$\nabla^i(\lambda^{-2}w_i) = 0\,,$$

$$\lambda(*_\gamma dw)^i = -2\lambda^{-1}T(g)^i_c\xi^c, \quad Ric(g) = G(T(g))\,.$$

Bibliography

[1] B.P. Abbott et al., *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence*, Phys. Rev. Lett. **116** (2016), 241103, arXiv:1606.04855 [gr-qc].

[2] ———, *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116** (2016), 061102, arXiv:1602.03837 [gr-qc].

[3] ———, *GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2*, Phys. Rev. Lett. **118** (2017), 221101.

[4] S. Akcay and R.A. Matzner, *Kerr-de Sitter Universe*, Class. Quantum Grav. **28** (2011), 085012.

[5] M. Alcubierre, B. Brugmann, D. Pollney, E. Seidel, and T. Takahashi, *Black hole excision for dynamic black holes*, Phys. Rev. **D64** (2001), 061501 (5pp.), arXiv:gr-qc/0104020.

[6] M. Alcubierre et al., *The 3d grazing collision of two black holes*, (2000), arXiv:gr-qc/0012079.

[7] S. Alexakis and V. Schlue, *Non-existence of time-periodic vacuum spacetimes*, Jour. Diff. Geom. **108** (2015), 1–62, arXiv:1504.04592. MR 3743702

[8] X. An and W.W.Y. Wong, *Warped product space-times*, Class. Quantum Grav. **35** (2018), 025011, 25 pp., arXiv:1707.01483 [gr-qc]. MR 3755944

[9] M.T. Anderson and P.T. Chruściel, *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions*, Commun. Math. Phys. **260** (2005), 557–577, arXiv:gr-qc/0412020. MR MR2183957

[10] L. Andersson and P.T. Chruściel, *On asymptotic behavior of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissert. Math. **355** (1996), 1–100 (English). MR MR1405962 (97e:58217)

[11] L. Andersson and R. Howard, *Rigidity results for Robertson–Walker and related space-times*, preprint (1997).

[12] J.G. Baker, J. Centrella, D.-Il Choi, M. Koppitz, and J. van Meter, *Gravitational wave extraction from an inspiraling configuration of merging black holes*, Phys. Rev. Lett. **96** (2006), 111102, arXiv:gr-qc/0511103.

[13] C. Barceló, S. Liberati, and M. Visser, *Analog gravity from field theory normal modes?*, Class. Quantum Grav. **18** (2001), 3595–3610, arXiv:gr-qc/0104001.

[14] J. Bardeen and G.T. Horowitz, *Extreme Kerr throat geometry: a vacuum analog of* $\mathrm{AdS}_2 \times S^2$, Phys. Rev. D (3) **60** (1999), 104030, 10, arXiv:hep-th/9905099. MR MR1757631 (2001f:83086)

[15] J.M. Bardeen, B. Carter, and S.W. Hawking, *The four laws of black hole mechanics*, Commun. Math. Phys. **31** (1973), 161–170.

[16] R. Bartnik and A. Norton, *NQS Einstein solver*, `http://beth.ise.canberra.edu.au/mathstat/StaffPages/Robert2.htm`.

[17] R. Beig and P.T. Chruściel, *Killing vectors in asymptotically flat space-times: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, Jour. Math. Phys. **37** (1996), 1939–1961, arXiv:gr-qc/9510015.

[18] ______, *The isometry groups of asymptotically flat, asymptotically empty space-times with timelike ADM four–momentum*, Commun. Math. Phys. **188** (1997), 585–597, arXiv:gr-qc/9610034.

[19] ______, *Killing Initial Data*, Class. Quantum Grav. **14** (1997), A83–A92, A special issue in honour of Andrzej Trautman on the occasion of his 64th Birthday, J.Tafel, editor. MR MR1691888 (2000c:83011)

[20] ______, *The asymptotics of stationary electro-vacuum metrics in odd space-time dimensions*, Class. Quantum Grav. **24** (2007), 867–874. MR MR2297271

[21] R. Beig and W. Simon, *Proof of a multipole conjecture due to Geroch*, Commun. Math. Phys. **78** (1980), 75–82.

[22] S. Benenti, *Computing curvature without Christoffel symbols*, SPT 2007—Symmetry and perturbation theory, World Sci. Publ., Hackensack, NJ, 2008, pp. 37–44. MR 2401901 (2009h:53035)

[23] B. Berger, P.T. Chruściel, J. Isenberg, and V. Moncrief, *Global foliations of vacuum space-times with T^2 isometry*, Ann. Phys. (NY) **260** (1997), 117–148.

[24] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 10, Springer Verlag, Berlin, New York, Heidelberg, 1987.

[25] H.R. Beyer and I. Craciun, *On a new symmetry of the solutions of the wave equation in the background of a kerr black hole*, (2006), arXiv:gr-qc/0607070.

[26] J. Bičák, M. Scholtz, and P. Tod, *On asymptotically flat solutions of Einstein's equations periodic in time I. Vacuum and electrovacuum solutions*, Class. Quantum Grav. **27** (2010), 055007, arXiv:1003.3402. MR 2594673

[27] G. D. Birkhoff, *Relativity and modern physics*, Harvard UP, 1923.

[28] D. Birmingham, *Topological black holes in anti-de Sitter space*, Class. Quantum Grav. **16** (1999), 1197–1205, arXiv:hep-th/9808032. MR MR1696149 (2000c:83062)

[29] N.T. Bishop, R. Gomez, L. Lehner, M. Maharaj, and J. Winicour, *High-powered gravitational news*, Phys. Rev. **D56** (1997), 6298–6309, arXiv:gr-qc/9708065.

[30] J. Bičák and J. Podolský, *The global structure of Robinson-Trautman radiative space-times with cosmological constant*, Phys. Rev. D **55** (1996), 1985–1993, arXiv:gr-qc/9901018.

[31] J. Bičák and B. Schmidt, *Asymptotically flat radiative spacetimes with boost–rotation symmetry: The general structure*, Phys. Rev. **D40** (1989), 1827–1853.

[32] R.H. Boyer and R.W. Lindquist, *Maximal analytic extension of the Kerr metric*, Jour. Math. Phys. **8** (1967), 265–281. MR MR0218093 (36 #1182)

[33] S. Brandt et al., *Grazing collisions of black holes via the excision of singularities*, Phys. Rev. Lett. **85** (2000), 5496–5499, arXiv:gr-qc/0009047.

[34] R. Brout, S. Massar, R. Parentani, and P. Spindel, *A primer for black hole quantum physics*, Phys. Rept. **260** (1995), 329–454.

[35] S.F. Browdy and G.J. Galloway, *Topological censorship and the topology of black holes.*, Jour. Math. Phys. **36** (1995), 4952–4961.

[36] P. Budinich and A. Trautman, *The spinorial chessboard*, Trieste Notes in Physics, Springer-Verlag, Berlin, 1988. MR MR954184 (90b:15020)

[37] G. Bunting and A.K.M. Masood–ul–Alam, *Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time*, Gen. Rel. Grav. **19** (1987), 147–154.

[38] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR 645743 (83i:53088)

[39] G.N. Candlish and H.S. Reall, *On the smoothness of static multi-black hole solutions of higher-dimensional einstein–maxwell theory*, (2007), arXiv:0707.4420 [gr-qc].

[40] P.R. Capelo, *Astrophysical black holes*, (2018), arXiv:1807.06014 [astro-ph.HE].

[41] B.J. Carr, *Primordial black holes as a probe of the early universe and a varying gravitational constant*, arXiv:astro-ph/0102390.

[42] B. Carter, *Complete analytic extension of the symmetry axis of Kerr's solution of Einstein's equations*, Phys. Rev. (2) **141** (1966), 1242–1247. MR 0205651 (34 #5477)

[43] ———, *Global structure of the Kerr family of gravitational fields*, Phys. Rev. **174** (1968), 1559–1571.

[44] ———, *Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations*, Commun. Math. Phys. **10** (1968), 280–310. MR 0239841 (39 #1198)

[45] ———, *Black hole equilibrium states*, Black Holes (C. de Witt and B. de Witt, eds.), Gordon & Breach, New York, London, Paris, 1973, Proceedings of the Les Houches Summer School.

[46] ———, *Mathematical foundation of the theory of relativistic stellar and black hole configurations*, Gravitation and Astrophysics (B. Carter and J.B. Hartle, eds.), Plenum Press, New York, 1987.

[47] ———, *Has the black hole equilibrium problem been solved?*, in Proc. of the 8th Marcel Grossmann Meeting on Relativistic Astrophysics - MG 8, Jerusalem, Israel, 22 - 27 June 1997, T. Piran, Ed., World Scientific, 1999, arXiv:gr-qc/9712038.

[48] A. Celotti, J.C. Miller, and D.W. Sciama, *Astrophysical evidence for the existence of black holes*, Class. Quantum Grav. **16** (1999), A3–A21, arXiv:astro-ph/9912186.

[49] J. Chaname, A. Gould, and J. Miralda-Escude, *Microlensing by the cluster of black holes around Sgr A^**, (2001), arXiv:astro-ph/0102481.

[50] S. Chandrasekhar, *Ellipsoidal figures of equilibrium*, Dover Publ., New York, 1969.

[51] ———, *The mathematical theory of black holes*, Oxford UP, 1984.

[52] Y. Choquet-Bruhat, P.T. Chruściel, and J. Loizelet, *Global solutions of the Einstein–Maxwell equations in higher dimension*, Class. Quantum Grav. (2006), no. 24, 7383–7394, arXiv:gr-qc/0608108. MR 2279722 (2008i:83022)

[53] D. Christodoulou, *Examples of naked singularity formation in the gravitational collapse of a scalar field*, Ann. Math. **140** (1994), 607–653.

[54] D. Christodoulou and S. Klainermann, *Nonlinear stability of Minkowski space*, Princeton University Press, Princeton, 1993.

[55] Piotr T. Chruściel and Gregory J. Galloway, *Horizons non-differentiable on a dense set*, Commun. Math. Phys. **193** (1998), no. 2, 449–470. MR 99c:83066

[56] Piotr T. Chruściel and Alan D. Rendall, *Strong cosmic censorship in vacuum space-times with compact, locally homogeneous Cauchy surfaces*, Ann. Physics **242** (1995), no. 2, 349–385.

[57] P.T. Chruściel, *On space-times with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *symmetric compact Cauchy surfaces*, Ann. Phys. **202** (1990), 100–150. MR MR1067565 (91h:83007)

[58] ———, *Semi-global existence and convergence of solutions of the Robinson-Trautman (2-Dimensional Calabi) equation*, Commun. Math. Phys. **137** (1991), 289–313.

[59] ———, *On the global structure of Robinson–Trautman space–times*, Proc. Roy. Soc. London **A 436** (1992), 299–316.

[60] ———, *On completeness of orbits of Killing vector fields*, Class. Quantum Grav. **10** (1993), 2091–2101, arXiv:gr-qc/9304029.

[61] ———, *"No Hair" Theorems – folklore, conjectures, results*, Differential Geometry and Mathematical Physics (J. Beem and K.L. Duggal, eds.), Cont. Math., vol. 170, AMS, Providence, 1994, pp. 23–49, arXiv:gr–qc/9402032.

[62] ———, *Uniqueness of black holes revisited*, Helv. Phys. Acta **69** (1996), 529–552, Proceedings of Journées Relativistes 96 (N. Straumann, Ph. Jetzer and G. Lavrelashvili, eds), arXiv:gr-qc/9610010.

[63] ———, *On rigidity of analytic black holes*, Commun. Math. Phys. **189** (1997), 1–7, arXiv:gr-qc/9610011.

[64] ———, *The classification of static vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 661–687, *Corrigendum* in arXiv:gr-qc/9809088v2.

[65] ———, *On higher dimensional black holes with abelian isometry group*, Jour. Math. Phys. **50** (2008), 052501 (21 pp.), arXiv:0812.3424 [gr-qc].

[66] ———, *Conformal boundary extensions of Lorentzian manifolds*, Jour. Diff. Geom. **84** (2010), 19–44, arXiv:gr-qc/0606101. MR 2629508 (2011j:53131)

[67] ———, *Elements of causality theory*, (2011), arXiv:1110.6706 [gr-qc].

[68] P.T. Chruściel and J. Cortier, *Maximal analytic extensions of the Emparan-Reall black ring*, Jour. Diff. Geom. **85** (2010), no. 3, 425–459, arXiv:0807.2309 [gr-qc]. MR 2739809

[69] P.T. Chruściel, J. Cortier, and A. Garcia-Parrado, *On the global structure of the Pomeransky-Senkov black holes*, **14** (2009), 1779–1856, arXiv:0911.0802 [gr-qc]. MR 2872470 (2012m:83032)

[70] P.T. Chruściel, J. Corvino, and J. Isenberg, *Construction of N-body initial data sets in general relativity*, Commun. Math. Phys. **304** (2010), 637–647 (English), arXiv:0909.1101 [gr-qc]. MR 2794541

[71] P.T. Chruściel and J. Lopes Costa, *On uniqueness of stationary black holes*, Astérisque **321** (2008), 195–265, arXiv:0806.0016v2 [gr-qc].

[72] P.T. Chruściel, J. Lopes Costa, and M. Heusler, *Stationary Black Holes: Uniqueness and Beyond*, Living Rev. Rel. **15** (2012), 7, arXiv:1205.6112 [gr-qc].

[73] P.T. Chruściel, E. Delay, G. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Annales Henri Poincaré **2** (2001), 109–178, arXiv:gr-qc/0001003. MR MR1823836 (2002e:83045)

[74] P.T. Chruściel, G. Galloway, and D. Solis, *Topological censorship for Kaluza-Klein spacetimes*, Ann. H. Poincaré **10** (2009), 893–912, arXiv:0808.3233 [gr-qc]. MR MR2533875

[75] P.T. Chruściel and G.J. Galloway, *Uniqueness of static black holes without analyticity*, Class. Quantum Grav. **27** (2010), 152001, 6, arXiv:1004.0513 [gr-qc]. MR 2659234

[76] P.T. Chruściel and J. Grant, *On Lorentzian causality with continuous metrics*, Class. Quantum Grav. **29** (2012), 145001, arXiv:1110.0400v2 [gr-qc].

[77] P.T. Chruściel and M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), 231–264, arXiv:dg-ga/0110035. MR MR2038048 (2005d:53052)

[78] P.T. Chruściel and J. Isenberg, *Non–isometric vacuum extensions of vacuum maximal globally hyperbolic space–times*, Phys. Rev. D (3) **48** (1993), 1616–1628. MR MR1236815 (94f:83007)

[79] P.T. Chruściel, J. Isenberg, and V. Moncrief, *Strong cosmic censorship in polarized Gowdy space–times*, Class. Quantum Grav. **7** (1990), 1671–1680.

[80] P.T. Chruściel, J. Jezierski, and J. Kijowski, *The Hamiltonian mass of asymptotically Schwarzschild-de Sitter space-times*, Phys. Rev. **D87** (2013), 124015 (11 pp.), arXiv:1305.1014 [gr-qc].

[81] P.T. Chruściel, M.A.H. MacCallum, and D. Singleton, *Gravitational waves in general relativity. XIV: Bondi expansions and the "polyhomogeneity" of Scri*, Philos. Trans. Roy. Soc. London Ser. A **350** (1995), 113–141, arXiv:gr-qc/9305021. MR MR1325206 (97f:83025)

[82] P.T. Chruściel and D. Maerten, *Killing vectors in asymptotically flat space-times: II. Asymptotically translational Killing vectors and the rigid positive energy theorem in higher dimensions*, Jour. Math. Phys. **47** (2006), 022502, 10 pp., arXiv:gr-qc/0512042. MR MR2208148 (2007b:83054)

[83] P.T. Chruściel and R. Mazzeo, *On "many-black-hole" vacuum space-times*, Class. Quantum Grav. **20** (2003), 729–754, arXiv:gr-qc/0210103. MR 1959399 (2004b:83061)

[84] P.T. Chruściel and N.S. Nadirashvili, *All electrovacuum Majumdar–Papapetrou space-times with non–singular black holes*, Class. Quantum Grav. **12** (1995), L17–L23, arXiv:gr-qc/9412044.

[85] P.T. Chruściel and G. Nagy, *The Hamiltonian mass of asymptotically anti-de Sitter space-times*, Class. Quantum Grav. **18** (2001), L61–L68, hep-th/0011270.

[86] P.T. Chruściel and L. Nguyen, *A uniqueness theorem for degenerate Kerr-Newman black holes*, Ann. H. Poincaré **11** (2010), 585–609, arXiv:1002.1737 [gr-qc]. MR 2677740 (2011i:53123)

[87] P.T. Chruściel, C.R. Ölz, and S.J. Szybka, *Space-time diagrammatics*, Phys. Rev. D **86** (2012), 124041, pp. 20, arXiv:1211.1718.

[88] P.T. Chruściel and T.-T. Paetz, *Light-cone initial data and smoothness of Scri. I. Formalism and results*, Ann. H. Poincaré **16** (2015), 2131–2162, arXiv:1403.3558 [gr-qc]. MR 3383324

[89] P.T. Chruściel, H.S. Reall, and K.P. Tod, *On Israel-Wilson-Perjès black holes*, Class. Quantum Grav. **23** (2006), 2519–2540, arXiv:gr-qc/0512116. MR MR2215078

[90] ______, *On non-existence of static vacuum black holes with degenerate components of the event horizon*, Class. Quantum Grav. **23** (2006), 549–554, arXiv:gr-qc/0512041. MR MR2196372 (2007b:83090)

[91] P.T. Chruściel and D.B. Singleton, *Nonsmoothness of event horizons of Robinson-Trautman black holes*, Commun. Math. Phys. **147** (1992), 137–162. MR 1171763 (93i:83059)

[92] P.T. Chruściel and S. Szybka, *Stable causality of the Pomeransky-Senkov black holes*, Adv. Theor. Math. Phys. **15** (2011), 175–178, arXiv:1010.0213 [hep-th].

[93] P.T. Chruściel, S.J. Szybka, and P. Tod, *Towards a classification of near-horizons geometries*, Class. Quantum Grav. **35** (2017), 015002, 14 pp., arXiv:1707.01118 [gr-qc]. MR 3747353

[94] P.T. Chruściel and K.P. Tod, *The classification of static electro-vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior*, Commun. Math. Phys. **271** (2007), 577–589. MR MR2291788

[95] P.T. Chruściel and R. Tagné Wafo, *Solutions of quasi-linear wave equations polyhomogeneous at null infinity in high dimensions*, Jour. Hyperb. Diff.Ėqu. **8** (2011), 269–346, arXiv:1010.2387 [gr-qc]. MR 2812145 (2012h:58036)

[96] P.T. Chruściel and R.M. Wald, *On the topology of stationary black holes*, Class. Quantum Grav. **11** (1994), L147–152, arXiv:gr–qc/9410004. MR MR1307013 (95j:83080)

[97] J. Creswell, S. von Hausegger, A.D. Jackson, H. Liu, and P. Naselsky, *On the time lags of the LIGO signals*, (2017).

[98] E. Czuchry, J. Jezierski, and J. Kijowski, *Dynamics of a gravitational field within a wave front and thermodynamics of black holes*, Phys. Rev. D **70** (2004), 124010, 14, arXiv:gr-qc/0412042. MR 2124700 (2005k:83071)

[99] M. Dafermos, G. Holzegel, and I. Rodnianski, *A scattering theory construction of dynamical vacuum black holes*, (2013), arXiv:1306.5364 [gr-qc].

[100] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor, *The nonlinear stability of the Schwarzschild solution to gravitational perturbations*, (2018), in preparation.

[101] M. Dafermos and I. Rodnianski, *Lectures on black holes and linear waves*, (2008), arXiv:0811.0354 [gr-qc].

[102] S. Dain, *Initial data for stationary spacetimes near spacelike infinity*, Class. Quantum Grav. **18** (2001), 4329–4338, arXiv:gr-qc/0107018.

[103] ______, *A variational principle for stationary, axisymmetric solutions of Einstein's equations*, Class. Quantum Grav. **23** (2006), 6857–6871, arXiv:gr-qc/0508061. MR MR2273525

[104] T. Damour and B. Schmidt, *Reliability of perturbation theory in general relativity*, Jour. Math. Phys. **31** (1990), 2441–2453.

[105] S.R. Das and S.D. Mathur, *The quantum physics of black holes: Results from string theory*, Ann. Rev. Nucl. Part. Sci. **50** (2000), 153–206, arXiv:gr-qc/0105063.

[106] M. Demiański, *Some new solutions of the Einstein equations of astrophysical interest*, Acta Astronomica **23** (1973), 197–231.

[107] S. Deser and Orit Levin, *Accelerated detectors and temperature in (anti-) de Sitter spaces*, Class. Quantum Grav. **14** (1997), L163–L168. MR 1477799 (99d:83050)

[108] ______, *Equivalence of Hawking and Unruh temperatures and entropies through flat space embeddings*, Class. Quantum Grav. **15** (1998), L85–L87. MR 1663546 (99i:83046)

[109] ______, *Mapping Hawking into Unruh thermal properties*, Phys. Rev. D **59** (1999), 064004, 7. MR 1678933 (2000a:83059)

[110] O.J.C. Dias, P. Figueras, R. Monteiro, H.S. Reall, and J.E. Santos, *An instability of higher-dimensional rotating black holes*, JHEP **1005** (2010), 076, 1001.4527. MR 2721129 (2011g:83060)

[111] R. Emparan, *Rotating circular strings, and infinite non-uniqueness of black rings*, JHEP **03** (2004), 064.

[112] R. Emparan and H.S. Reall, *A rotating black ring in five dimensions*, Phys. Rev. Lett. **88** (2002), 101101, arXiv:hep-th/0110260.

[113] ______, *Black rings*, Class. Quantum Grav. **23** (2006), R169–R197, arXiv:hep-th/0608012.

[114] ______, *Black Holes in Higher Dimensions*, Living Rev. Rel. **11** (2008), 6, arXiv:0801.3471 [hep-th].

[115] A. Fathi and A. Siconolfi, *On smooth time functions*, Math. Proc. Camb. Phil. Soc. **152** (2012), 303–339. MR 2887877

[116] L. Ferrarese and H. Ford, *Supermassive black holes in galactic nuclei: Past, present and future research*, Space Sci.Rev. **116** (2005), 523–624, arXiv:astro-ph/0411247.

[117] P. Figueras, K. Murata, and H.S. Reall, *Black hole instabilities and local Penrose inequalities*, Class. Quantum Grav. **28** (2011), 225030, 30, arXiv:1107.5785 [gr-qc]. MR 2871933 (2012m:83070)

[118] J.L. Friedman, K. Schleich, and D.M. Witt, *Topological censorship*, Phys. Rev. Lett. **71** (1993), 1486–1489, erratum **75** (1995) 1872.

[119] H. Friedrich, *Cauchy problems for the conformal vacuum field equations in general relativity*, Commun. Math. Phys. **91** (1983), 445–472 (English). MR 727195

[120] ______, *On the existence of n–geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Commun. Math. Phys. **107** (1986), 587–609.

[121] ———, *Einstein's equation and geometric asymptotics*, Gravitation and Relativity: At the turn of the Millennium, N. Dahdich and J. Narlikar (eds.), IUCAA, Pune, 1998, Proceedings of GR15, pp. 153–176.

[122] ———, *Smoothness at null infinity and the structure of initial data*, The Einstein Equations and the Large Scale Behavior of Gravitational Fields (P.T. Chruściel and H. Friedrich, eds.), Birkhäuser, Basel, 2004, pp. 121–203, arXiv:gr–qc/0304003.

[123] H. Friedrich and B.G. Schmidt, *Conformal geodesics in general relativity*, Proc. Roy. Soc. London Ser. A **414** (1987), 171–195. MR 89b:83041

[124] C. Fronsdal, *Completion and embedding of the Schwarzschild solution*, Phys. Rev. (2) **116** (1959), 778–781. MR MR0110524 (22 #1402)

[125] G.J. Galloway, *On the topology of the domain of outer communication*, Class. Quantum Grav. **12** (1995), L99–L101.

[126] ———, *A "finite infinity" version of the FSW topological censorship*, Class. Quantum Grav. **13** (1996), 1471–1478. MR MR1397128 (97h:83065)

[127] ———, *Maximum principles for null hypersurfaces and null splitting theorems*, Ann. H. Poincaré **1** (2000), 543–567. MR MR1777311 (2002b:53052)

[128] G.J. Galloway, K. Schleich, D.M. Witt, and E. Woolgar, *The AdS/CFT correspondence conjecture and topological censorship*, Phys. Lett. **B505** (2001), 255–262, arXiv:hep-th/9912119.

[129] G.J. Galloway and E. Woolgar, *The cosmic censor forbids naked topology*, Class. Quantum Grav. **14** (1996), L1–L7, arXiv:gr-qc/9609007.

[130] D. Garfinkle and S.G. Harris, *Ricci fall-off in static and stationary, globally hyperbolic, non-singular space-times*, Class. Quantum Grav. **14** (1997), 139–151, arXiv:gr-qc/9511050.

[131] C. Gérard, *Rotating Kaluza-Klein monopoles and dyons*, Phys. Lett. A **118** (1986), 11–13. MR 859673

[132] R. Geroch, *Asymptotic structure of space-time*, Asymptotic structure of spacetime (Proc. Sympos., Univ. Cincinnati, Cincinnati, Ohio, 1976), Plenum, New York, 1977, pp. 1–105. MR MR0484240 (58 #4166)

[133] R. Geroch and G. Horowitz, *Asymptotically simple does not imply asymptotically Minkowskian*, Phys. Rev. Lett. **40** (1978), 203–206.

[134] R. Geroch, E.H. Kronheimer, and R. Penrose, *Ideal points in space-time.*, Proc. Roy. Soc. London Ser. A **327** (1972), 545–567. MR MR0316035 (47 #4583)

[135] A.M. Ghez, S. Salim, N.N. Weinberg, J.R. Lu, T. Do, et al., *Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits*, Astrophys. Jour. **689** (2008), 1044–1062, arXiv:0808.2870 [astro-ph].

[136] G. W. Gibbons, H. Lü, Don N. Page, and C. N. Pope, *The general Kerr-de Sitter metrics in all dimensions*, Jour. Geom. Phys. **53** (2005), 49–73, arXiv:hep-th/0404008. MR 2102049 (2005i:53056)

[137] G.W. Gibbons and S.W. Hawking (eds.), *Euclidean quantum gravity*, World Scientific Publishing Co., Singapore, 1993.

[138] G.W. Gibbons and C.A.R. Herdeiro, *Supersymmetric rotating black holes and causality violation*, Class. Quantum Grav. **16** (1999), 3619–3652, arXiv:hep-th/9906098.

[139] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, et al., *Monitoring stellar orbits around the Massive Black Hole in the Galactic Center*, Astrophys. Jour. **692** (2009), 1075–1109, arXiv:0810.4674 [astro-ph].

[140] R. Gomez et al., *Stable characteristic evolution of generic 3-dimensional single-black-hole space-times*, Phys. Rev. Lett. **80** (1998), 3915–3918, arXiv:gr-qc/9801069.

[141] R.H. Gowdy, *Gravitational waves in closed universes*, Phys. Rev. Lett. **27** (1971), 826.

[142] M. Graf, J.D.E. Grant, M. Kunzinger, and R. Steinbauer, *The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics*, (2017), arXiv:1706.08426 [math-ph].

[143] P. Grandclément, E. Gourgoulhon, and S. Bonazzola, *Binary black holes in circular orbits. II. Numerical methods and first results*, (2001), arXiv:gr-qc/0106016.

[144] J.E. Greene and L.C. Ho, *Active galactic nuclei with candidate intermediate - mass black holes*, Astrophys. Jour. **610** (2004), 722–736, arXiv:astro-ph/0404110.

[145] A. Gullstrand, *Allegemeine Lösung des statischen Einkörper-Problems in der Einsteinshen Gravitations Theorie*, Arkiv. Mat. Astron. Fys. **16** (1922), 1–15.

[146] J.B. Hartle and S.W. Hawking, *Solutions of the Einstein–Maxwell equations with many black holes*, Commun. Math. Phys. **26** (1972), 87–101.

[147] S.W. Hawking and G.F.R. Ellis, *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973, Cambridge Monographs on Mathematical Physics, No. 1. MR MR0424186 (54 #12154)

[148] J. Hempel, *3–manifolds*, Princeton University Press, Princeton, 1976, Annals of Mathematics Studies No 86.

[149] M. Henneaux and C. Teitelboim, *Asymptotically anti–de Sitter spaces*, Commun. Math. Phys. **98** (1985), 391–424. MR 86f:83030

[150] M. Heusler, *Black hole uniqueness theorems*, Cambridge University Press, Cambridge, 1996.

[151] P. Hintz and A. Vasy, *A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric*, (2017), arXiv:1711.00195 [math.AP].

[152] S. Hollands and A. Ishibashi, *Instabilities of (near) extremal rotating black holes in higher dimensions*, (2014), arXiv:1408.0801 [hep-th].

[153] S. Hollands and S. Yazadjiev, *A uniqueness theorem for stationary Kaluza-Klein black holes*, Commun. Math. Phys. **283** (2008), 749–768, arXiv:0812.3036 [gr-qc]. MR MR2434746 (2009e:83093)

[154] ______, *Uniqueness theorem for 5-dimensional black holes with two axial Killing fields*, Commun. Math. Phys. **283** (2008), 749–768, arXiv:0707.2775 [gr-qc]. MR MR2434746

[155] G.T. Horowitz, *Quantum states of black holes*, R.M. Wald (ed.), Black holes and relativistic stars. Symposium held in Chicago, IL, USA, December 14-15, 1996. Univ. of Chicago Press, Chicago, Ill., pp. 241–266, arXiv:gr-qc/9704072.

[156] G.T. Horowitz and R.C. Myers, *The AdS/CFT correspondence and a new positive energy conjecture for general relativity*, Phys. Rev. **D59** (1998), 026005 (12 pp.), arXiv:hep-th/9808079.

[157] M. Hörzinger, *Black holes in higher dimensions*, Ph.D. thesis, Vienna, 2018.

[158] L. Iorio, *Constraining the angular momentum of the Sun with planetary orbital motions and general relativity*, Solar Phys. **281** (2012), 815–826, arXiv:1112.4168 [gr-qc].

[159] W. Israel, *New interpretation of the extended Schwarzschild manifold*, Phys. Rev. **143** (1966), 1016–1021.

[160] ______, *Dark stars: The evolution of an idea*, S.W. Hawking and W. Israel (eds.), Three hundred years of gravitation. Cambridge: Cambridge University Press, pp. 199-276, 1987.

[161] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. **D50** (1994), 846–864. MR 1288716 (95e:83060)

[162] T. Jacobson, *Trans-planckian redshifts and the substance of the space- time river*, Prog. Theor. Phys. Suppl. **136** (1999), 1–17, hep-th/0001085.

[163] T. Jacobson and S. Venkatarami, *Topology of event horizons and topological censorship*, Class. Quantum Grav. **12** (1995), 1055–1061.

[164] J.T. Jebsen, *On the general spherically symmetric solutions of Einstein's gravitational equations in vacuo*, Gen. Rel. Grav. **37** (2005), 2253–2259, translation of Ark. Mat. Ast. Fys. (Stockholm) 15, nr.18 (1921).

[165] J. Jezierski and B. Kamiński, *Towards uniqueness of degenerate axially symmetric Killing horizon*, Gen. Rel. Grav. **45** (2013), 987–1004, arXiv:1206.5136 [gr-qc]. MR 3046208

[166] D. Kastor and J. Traschen, *Cosmological multi–black hole solutions*, Phys. Rev. **D47** (1993), 5370–5375, hep-th/9212035.

[167] B.S. Kay and R.M. Wald, *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasi-free states on space-times with a bifurcate horizon*, Phys. Rep. **207** (1991), 49–136.

[168] J.D.P. Kenney and E.E. Yale, *Hubble space telescope imaging of bipolar nuclear shells in the disturbed Virgo Cluster Galaxy NGC 4438*, (2000), arXiv:astro-ph/0005052.

[169] L.E. Kidder, M.A. Scheel, and S.A. Teukolsky, *Extending the lifetime of 3d black hole computations with a new hyperbolic system of evolution equations*, Phys. Rev. **D64** (2001), 064017, arXiv:gr-qc/0105031.

[170] S. Klainerman and J. Szeftel, *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*, (2017), arXiv:1711.07597 [gr-qc].

[171] T. Klösch and T. Strobl, *Explicit global coordinates for Schwarzschild and Reissner–Nordström solutions*, Class. Quantum Grav. **13** (1996), 1191–1199.

[172] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience Publishers, New York, 1963.

[173] J. Kormendy and K. Gebhardt, *Supermassive black holes in galactic nuclei*, (2001), arXiv:astro-ph/0105230.

[174] F. Kottler, *Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie*, Annalen der Physik **56** (1918), 401–462.

[175] J.A. Valiente Kroon, *Can one detect a non-smooth null infinity?*, Class. Quantum Grav. **18** (2001), 4311–4316.

[176] M.D. Kruskal, *Maximal extension of Schwarzschild metric*, Phys. Rev. (2) **119** (1960), 1743–1745. MR MR0115757 (22 #6555)

[177] W. Kundt and M. Trümper, *Orthogonal decomposition of axi-symmetric stationary space-times*, Z. Physik **192** (1966), 419–422. MR MR0197165 (33 #5347)

[178] M. Kunzinger, R. Steinbauer, M. Stojkovic, and J.A. Vickers, *Hawking's singularity theorem for $C^{1,1}$-metrics*, Class. Quantum Grav. **32** (2015), 075012, arXiv:1411.4689 [gr-qc].

[179] M. Kunzinger, R. Steinbauer, and J.A. Vickers, *The Penrose singularity theorem in regularity $C^{1,1}$*, Class. Quantum Grav. **32** (2015), 155010, arXiv:1502.00287 [gr-qc].

[180] D.N. Kupeli, *On null submanifolds in spacetimes*, Geom. Dedicata **23** (1987), 33–51.

[181] K. Lake, `http://grtensor.org/blackhole`.

[182] ______, *A maximally extended, explicit and regular covering of the Schwarzschild - de Sitter vacua in arbitrary dimension. I. General formulae*, (2005), arXiv:gr-qc/0507031.

[183] L. Lehner, *Numerical relativity: A review*, Class. Quantum Grav. **18** (2001), R25–R86, arXiv:gr-qc/0106072.

[184] G. Lemaître, *L'univers en expansion*, Ann. Soc. Sci. (Bruxelles) **A53** (1933), 51–85.

[185] O. Lengard, *Solutions of the Einstein's equation, wave maps, and semilinear waves in the radiation regime*, Ph.D. thesis, Université de Tours, 2001, `http://homepage.univie.ac.at/piotr.chrusciel/papers/batz/ls.html`.

[186] U. Leonhardt and P. Piwnicki, *Relativistic effects of light in moving media with extremely low group velocity*, Phys. Rev. Lett. **84** (2000), 822–825.

[187] J. Lewandowski and T. Pawłowski, *Quasi-local rotating black holes in higher dimension: geometry*, Class. Quantum Grav. **22** (2005), 1573–1598, arXiv:gr-qc/0410146.

[188] H. Lindblad and I. Rodnianski, *Global existence for the Einstein vacuum equations in wave coordinates*, Commun. Math. Phys. **256** (2005), 43–110, arXiv:math.ap/0312479. MR MR2134337 (2006b:83020)

[189] D. Lovelock, *Divergence-free tensorial concomitants*, Aequationes Math. **4** (1970), 127–138. MR 0270285

[190] ______, *The Einstein tensor and its generalizations*, Jour. Math. Phys. **12** (1971), 498–501. MR 0275835 (43 #1588)

[191] ______, *The four-dimensionality of space and the Einstein tensor*, Jour. Math. Phys. **13** (1972), 874–876. MR 0297306

[192] B. Lukács, Z. Perjés, J. Porter, and Á. Sebestyén, *Lyapunov functional approach to radiative metrics*, Gen. Rel. Grav. **16** (1984), 691–701. MR 750380 (85m:83025)

[193] J.-P. Luminet, *Black holes: A general introduction.*, F.W. Hehl, C. Kiefer and R.J.K. Metzler (eds.), Black holes: theory and observation. Proceedings of the 179th W.E. Heraeus seminar, Bad Honnef, Germany, August 18-22, 1997. Springer Lect. Notes Phys. 514, pp. 3-34 , arXiv:astro-ph/9804006.

[194] F.D. Macchetto, *Supermassive Black Holes and Galaxy Morphology*, Astrophys. Space Sci. **269** (1999), 269–291, arXiv:astro-ph/9910089.

[195] G.M. Madejski, *Black holes in active galactic nuclei: observations*, (1999), arXiv:astro-ph/9903141.

[196] J.P. Maillard, T. Paumard, S. R. Stolovy, and F. Rigaut, *The nature of the Galactic Center source IRS 13 revealed by high spatial resolution in the infrared*, Astron. Astrophys. **423** (2004), 155–167, arXiv:astro-ph/0404450.

[197] S.D. Majumdar, *A class of exact solutions of Einstein's field equations*, Phys. Rev. **72** (1947), 390–398.

[198] M. Mars and M. Reiris, *Global and uniqueness properties of stationary and static spacetimes with outer trapped surfaces*, Commun. Math. Phys. **322** (2013), 633–666, arXiv:1206.0607 [gr-qc]. MR 3077927

[199] J.M. Martín-García, *xAct: Efficient Tensor Computer Algebra*, `http://www.xact.es`.

[200] A.K.M. Masood–ul–Alam, *Uniqueness proof of static charged black holes revisited*, Class. Quantum Grav. **9** (1992), L53–L55.

[201] J.E. McClintock and R.A. Remillard, *Black Hole Binaries*, (2003), arXiv:astro-ph/0306213.

[202] R. Meinel, R.M. Ansorg, A. Kleinwächter, G. Neugebauer, and D. Petroff, *Relativistic figures of equilibrium*, Cambridge University Press, Cambridge, 2008. MR MR2441850 (2009g:83001)

[203] K. Menou, E. Quataert, and R. Narayan, *Astrophysical evidence for black hole event horizons*, Gravitation and Relativity: At the turn of the Millennium, N. Dahdich and J. Narlikar (eds.), IUCAA, Pune, 1998, Proceedings of GR15, pp. 43–65.

[204] D. Merritt and L. Ferrarese, *Relationship of black holes to bulges*, (2001), arXiv:astro-ph/0107134.

[205] E. Minguzzi and M. Sánchez, *The causal hierarchy of spacetimes*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 299–358. MR 2436235 (2010b:53128)

[206] J. Miralda-Escudé and A. Gould, *A cluster of black holes at the galactic center*, Astroph. Jour. **545** (2000), 847–853, arXiv:astro-ph/0003269.

[207] C.W. Misner, *Taub–NUT space as a counterexample to almost anything*, Relativity Theory and Astrophysics, AMS, Providence, Rhode Island, 1967, Lectures in Appl. Math., vol. 8, pp. 160–169.

[208] C.W. Misner, K. Thorne, and J.A. Wheeler, *Gravitation*, Freeman, San Fransisco, 1973.

[209] V. Moncrief, *Neighborhoods of Cauchy horizons in cosmological spacetimes with one Killing field*, Ann. Phys. **141** (1982), 83–103. MR MR666736 (83j:83005)

[210] V. Moncrief and J. Isenberg, *Symmetries of cosmological Cauchy horizons*, Commun. Math. Phys. **89** (1983), 387–413. MR 709474 (85c:83026)

[211] J.M. Morgan, L.J Greenhill, and J.R. Hernstein, *Observational evidence for massive black holes in the centers of active galaxies*, arXiv:astro-ph/0002085.

[212] A. Mueller, *Experimental evidence of black holes*, Proc. of Science **P2GC** (2006), 017, arXiv:astro-ph/0701228.

[213] R.C. Myers, *Higher dimensional black holes in compactified space-times*, Phys. Rev. **D35** (1987), 455–466.

[214] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, Ann. Phys. **172** (1986), 304–347.

[215] R. Narayan, M.R. Garcia, and J.E. McClintock, *X-ray Novae and the evidence for black hole event horizons*, (2001), arXiv:astro-ph/0107387.

[216] R. Narayan, I. Yi, and R. Mahadevan, *Explaining the spectrum of Sagittarius-A* with a model of an accreting black-hole*, Nature **374** (1995), 623.

[217] H. Nariai, *On a new cosmological solution of Einstein's field equations of gravitation*, Sci. Rep. Tohoku Univ. Ser. I. **35** (1951), 62–67. MR 14,1133f

[218] E. Newman, L. Tamburino, and T. Unti, *Empty-space generalization of the Schwarzschild metric*, Jour. Math. Phys. **4** (1963), 915–923. MR MR0152345 (27 #2325)

[219] J.-P. Nicolas, *Dirac fields on asymptotically flat space-times*, Dissertationes Math. (Rozprawy Mat.) **408** (2002), 1–85. MR 1 952 742

[220] K. Nomizu, *On local and global existence of Killing vector fields*, Ann. Math. **72** (1960), 105–120. MR MR0119172 (22 #9938)

[221] M. Novello, J.M. Salim, V.A. De Lorenci, and E. Elbaz, *Nonlinear electrodynamics can generate a closed spacelike path for photons*, Phys. Rev. **D63** (2001), 103516 (5 pp.).

[222] P. Nurowski and M. Randall, *Generalized Ricci solitons*, Jour. Geom. Anal. **26** (2016), 1280–1345, arXiv:1409.4179 [math.DG]. MR 3472837

[223] C.R. Oelz, *The global structure of kerr-de sitter metrics*, Master's thesis, Vienna University, 2013.

[224] C.R. Oelz and S.J. Szybka, *Conformal and projection diagrams in LaTeX*, (2013).

[225] B. O'Neill, *The geometry of Kerr black holes*, A K Peters, Ltd., Wellesley, MA, 1995. MR 1328643 (96c:83052)

[226] J.A. Orosz, J.E. McClintock, J.P. Aufdenberg, R.A. Remillard, M.J. Reid, R. Narayan, and L. Gou, *The Mass of the Black Hole in Cygnus X-1*, AstrophysJ̇. **742** (2011), 84, arXiv:1106.3689 [astro0ph.HE].

[227] P. Painlevé, *La mécanique classique el la theorie de la relativité*, C. R. Acad. Sci. (Paris) **173** (1921), 677–680.

[228] D.W. Pajerski and E.T. Newman, *Trapped surfaces and the development of singularities*, Jour. Math. Phys. **12** (1971), 1929–1937.

[229] A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge distribution*, Proc. Roy. Irish Acad. **A51** (1947), 191–204.

[230] ———, *Champs gravitationnels stationnaires à symmétrie axiale*, Ann. Inst. H. Poincaré **4** (1966), 83–105.

[231] L.K. Patel, R. Tikekar, and N. Dadhich, *Higher-dimensional analogue of McVittie solution*, Grav. Cosmol. **6** (2000), 335–336, arXiv:gr-qc/9909069.

[232] A.W. Peet, *TASI lectures on black holes in string theory*, Boulder 1999: Strings, branes and gravity (J. Harvey, S. Kachru, and E. Silverstein, eds.), World Scientific, Singapore, 2001, hep-th/0008241, pp. 353–433.

[233] N. Pelavas, N. Neary, and K. Lake, *Properties of the instantaneous ergo surface of a Kerr black hole*, Class. Quantum Grav. **18** (2001), 1319–1332, arXiv:gr-qc/0012052.

[234] R. Penrose, *Zero rest-mass fields including gravitation: Asymptotic behaviour*, Proc. Roy. Soc. London Ser. A **284** (1965), 159–203. MR MR0175590 (30 #5774)

[235] J. Podolsky and M. Ortaggio, *Robinson-Trautman spacetimes in higher dimensions*, Class. Quantum Grav. **23** (2006), 5785–5797, arXiv;gr-qc/0605136.

[236] A.A. Pomeransky and R.A. Senkov, *Black ring with two angular momenta*, (2006), hep-th/0612005.

[237] F. Pretorius, *Evolution of binary black hole spacetimes*, Phys. Rev. Lett. **95** (2005), 121101, arXiv:gr-qc/0507014.

[238] I. Rácz, *Space-time extensions. I*, Jour. Math. Phys. **34** (1993), 2448–2464. MR MR1219001 (94h:53094)

[239] I. Rácz and R.M. Wald, *Global extensions of space-times describing asymptotic final states of black holes*, Class. Quantum Grav. **13** (1996), 539–552, arXiv:gr-qc/9507055. MR MR1385315 (97a:83071)

[240] D. Rasheed, *The Rotating dyonic black holes of Kaluza-Klein theory*, Nucl. Phys. **B454** (1995), 379–401, arXiv:hep-th/9505038.

[241] M.J. Rees, *Astrophysical evidence for black holes*, R.M. Wald (ed.), Black holes and relativistic stars. Symposium held in Chicago, IL, USA, December 14-15, 1996. Univ. of Chicago Press, Chicago, Ill., pp. 79-101.

[242] ______, *Supermassive black holes: Their formation, and their prospects as probes of relativistic gravity*, Proceedings of ESO conference in honour of R. Giacconi on Black Holes in Binaries and Galactic Nuclei, 2000, arXiv:astro-ph/9912346.

[243] T. Regge and C. Teitelboim, *Role of surface integrals in the Hamiltonian formulation of general relativity*, Ann. Phys. **88** (1974), 286–318.

[244] A.D. Rendall, *Existence and asymptotic properties of global solutions of the Robinson-Trautman equation*, Classi. Quantum Grav. **5** (1988), 1339–1347. MR 964979 (89k:83009)

[245] H. Ringström, *Data at the moment of infinite expansion for polarized Gowdy*, Class. Quantum Grav. **22** (2005), 1647–1653.

[246] ______, *Strong cosmic censorship in T^3-Gowdy space-times*, Ann. of Math. (2) **170** (2009), 1181–1240. MR MR2600872

[247] I. Robinson and A. Trautman, *Spherical gravitational waves in general relativity*, Proc. Roy. Soc. Lond. A **270** (1962), 103–126.

[248] P. Ruback, *A new uniqueness theorem for charged black holes*, Class. Quantum Grav. **5** (1988), L155–L159.

[249] J. Sbierski, *The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian Geometry*, (2015), arXiv:1507.00601 [gr-qc].

[250] B.G. Schmidt, *Existence of solutions of the Robinson–Trautman equation and spatial infinity*, Gen. Rel. Grav. **20** (1988), 65–70.

[251] ______, *On the uniqueness of boundaries at infinity of asymptotically flat spacetimes*, Class. Quantum Grav. **8** (1991), 1491–1504. MR MR1122247 (92m:83037a)

[252] H. Schmidt, *A New conformal duality of spherically symmetric space-times*, Annalen Phys. **9SI** (2000), 158–159, arXiv:gr-qc/9905103.

[253] ______, *The tetralogy of birkhoff theorems*, General Relativity and Gravitation **45** (2013), no. 2, 395–410.

[254] J.A. Schouten, *Tensor Analysis for Physicists*, Oxford, Clarendon Press, 1951. MR 13,493e

[255] W. Simon, *Radiative Einstein-Maxwell spacetimes and 'no–hair' theorems*, Class. Quantum Grav. **9** (1992), 241–256.

[256] W. Simon and R. Beig, *The multipole structure of stationary space-times*, Jour. Math. Phys. **24** (1983), 1163–1171.

[257] D. Singleton, *On global existence and convergence of vacuum Robinson-Trautman solutions*, Class. Quantum Grav. **7** (1990), 1333–1343. MR 1064183 (91h:83023)

[258] The Hubble site of the Space Telescope Science Institute, `http://hubble.stsci.edu`.

[259] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2003 (2nd ed.). MR MR2003646 (2004h:83017)

[260] M. Struwe, *Curvature flows on surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), 247–274. MR 1991140 (2004f:53083)

[261] Gy. Szekeres, *On the singularities of a Riemannian manifold*, Gen. Rel. Grav. **34** (2002), 2001–2016, Reprinted from Publ. Math. Debrecen **7** (1960), 285–301 [MR0125541 (23 #A2842)]. MR MR1945497 (2003j:83073a)

[262] L.A. Tamburino and J.H. Winicour, *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150** (1966), 1039–1053.

[263] F.R. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, Nuovo Cimento (10) **27** (1963), 636–651. MR MR0148448 (26 #5955)

[264] A.H. Taub, *Empty spacetimes admitting a three parameter group of motions*, Ann. of Math. (2) **53** (1951), 472–490. MR MR0041565 (12,865b)

[265] K.P. Tod, *Analogues of the past horizon in the Robinson-Trautman metrics*, Class. Quantum Grav. **6** (1989), 1159–1163. MR 1005655

[266] A. Trautman, *Connections and the Dirac operator on spinor bundles*, Jour. Geom. Phys. **58** (2008), 238–252. MR MR2384313 (2008k:58077)

[267] J.-H. Treude and J.D.E. Grant, *Volume comparison for hypersurfaces in Lorentzian manifolds and singularity theorems*, Ann. Global Anal. Geom. **43** (2013), 233–251, arXiv:1201.4249 [gr-qc]. MR 3027611

[268] Z.I. Tsvetanov, M.G. Allen, H.C. Ford, and R.J. Harms, *Morphology of the nuclear disk in M87*, 1998, Proceedings of the M87 Workshop, Ringberg castle, Germany, 15-19 Sep 1997, arXiv:astro-ph/9803178.

[269] W.G. Unruh, *Dumb holes and the effects of high frequencies on black hole evaporation*, Phys. Rev. **D51** (1995), 2827–2838, arXiv:gr-qc/9409008.

[270] P. Urbański, *Analiza dla studentów fizyki*, Uniwersytet Warszawski, 1998 (in polish).

[271] R.M. Wald, *General relativity*, University of Chicago Press, Chicago, 1984.

[272] ———, *The thermodynamics of black holes*, Living Reviews **4** (2001), arXiv:gr-qc/9912119, URL `http://www.livingreviews.org/`.

[273] R. Waldmüller, *A flat manifold with no symmetries*, Experimental Math. **12** (2003), 71–77. MR 2002674

[274] M. Walker, *Bloc diagrams and the extension of timelike two–surfaces*, Jour. Math. Phys. **11** (1970), 2280–2286.

[275] G. Weinstein, *N-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations*, Commun. Part. Diff. Eqs. **21** (1996), 1389–1430.

[276] D.L. Welch, *On the smoothness of the horizons of multi - black hole solutions*, Phys. Rev. **D52** (1995), 985–991, arXiv:hep-th/9502146.

[277] H. Weyl, *On the Volume of Tubes*, Amer. Jour. Math. **61** (1939), 461–472. MR 1507388

[278] N.M.J. Woodhouse, *General relativity*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2007. MR MR2268691 (2007i:83003)

[279] E. Woolgar, *The rigid Horowitz-Myers conjecture*, Jour. High Energy Phys. (2016), 104, 26 pp., arXiv:1602.06197. MR 3650713

[280] S. Yelda et al., *Increasing the Scientific Return of Stellar Orbits at the Galactic Center*, (2010), arXiv:1002.1729.

[281] T. de Zeeuw, *Evidence for massive black holes in nearby galactic nuclei*, (2000), arXiv:astro-ph/0009249.

[282] B. Zumino, *Gravity theories in more than four dimensions*, Phys. Rep. **137** (1986), 109–114, New avenues in quantum theory and general relativity (Amalfi, 1984). MR 839214